

2019

# Modelling of Fatigue Crack Closure via the Concept of Plastic Inclusions

ALOUSTA, EBTESAM

<http://hdl.handle.net/10026.1/13437>

---

<http://dx.doi.org/10.24382/1023>

University of Plymouth

---

*All content in PEARL is protected by copyright law. Author manuscripts are made available in accordance with publisher policies. Please cite only the published version using the details provided on the item record or document. In the absence of an open licence (e.g. Creative Commons), permissions for further reuse of content should be sought from the publisher or author.*



**UNIVERSITY OF  
PLYMOUTH**

**Modelling of fatigue crack closure via the concept of  
plastic inclusions**

By

**EBTESAM ALOUSTA**

A thesis submitted to the University of Plymouth  
in partial fulfilment for the degree of

**DOCTOR OF PHILOSOPHY**

Schools of Computing and Mathematics

2019

## Copyright Statement

This copy of the thesis has been supplied on condition that anyone who consults it is understood to recognise that its copyright rests with its author and that no quotation from the thesis and no information derived from it may be published without the author's prior consent.

# Modelling of fatigue crack closure via the concept of plastic inclusions

Ebtesam Alousta

## **Abstract**

The phenomenon of closure is one of the most important phenomena that have been linked to the deeper understanding of fatigue cracks. Various methods have been employed to model the plastic zone around the crack tip which appears to give rise to the closure phenomena.

In this thesis, our goal is to model the plastic region near the crack tip by using a suitable adaptation of an Eshelby inclusion. To do this, the first task is to translate Eshelby's solution in terms of Muskhelishvili's complex potential functions for 2D elasticity and then solve these equations for a suitable shape of plastic inclusion. In this thesis, we have concentrated only on the case where the inclusion is a disc in front of the crack just touching the crack tip.

# Contents

Abstract .....	4
Contents .....	5
List of figure .....	9
Nomenclature .....	10
Acknowledgements.....	11
Author's declaration.....	12
Chapter 1 Introduction.....	13
1.1 Motivations and aims of the research.....	14
1.2 Description of the contents .....	15
Chapter 2 Background .....	16
2.1 Introduction.....	16
2.2 Fundamental theory of continuum mechanics .....	16
2.2.1 Stress and equilibrium .....	16
2.2.2 Strain and Compatibility .....	18
2.2.3 The Equilibrium equation written in terms of displacements.....	21
2.3 Linear elastic in 2D .....	26
2.3.1 Review of plane stress and plane strain field equations .....	26
2.4 The linear elastic field by using the Green's function.....	33

2.4.1 Equilibrium equation for an infinite body.....	33
2.4.2 Green's function in Fourier space.....	35
2.4.3 The elastic stress field in 2D.....	38
2.5 Principles of fracture mechanics and their implications for fatigue .....	67
2.5.1 Introduction.....	67
2.5.2 Linear elastic fracture mechanics.....	67
2.5.3 Crack tip stress analysis.....	69
2.6 Fundamentals of fatigue crack closure phenomenon.....	72
2.7 Description of the models for defining crack tip stress and displacement fields .....	73
2.8 Concluding comments .....	75
Chapter 3 Literature review.....	76
3.1 Introduction.....	76
3.2.1 Mechanisms and theoretical evaluation of crack closure.....	77
3.2.2 Techniques for experimental measurement of fatigue crack closure.....	79
3.3 Eshelby's inclusion and cracks.....	85
3.4 Conclusion.....	88
Chapter 4 Complex potential representation of problem.....	91
4.1 Introduction.....	91
4.2 Method of complex potential for plane elasticity (The Kolosov- Muskhelishvili formulas).....	91

4.2.1 Complex potential representation of the Airy stress function .....	91
4.2.2 The complex potential representation of the scalar components of stress and displacements induced by point forces.....	96
4.2.2.1 The complex potential representation of the scalar components of stress induced by point forces .....	96
4.2.2.2 The complex potential representation of the scalar components of displacements induced by point forces .....	103
4.3 Westergaard method for mode I (symmetric).....	106
4.4 Westergaard approach for Mode-II (anti-symmetric) problem.....	109
4.5 The elasticity solutions by using the Westergaard function method.....	110
4.6 A new Westergaard's function to model the shear stresses along the crack flanks.....	120
4.7 Concluding comments .....	131
Chapter 5 Classic Eshelby's inclusion .....	132
5.1 Introduction.....	132
5.2 Translation of Eshelby's solution in terms of Muskhelishvili's complex function approach to 2D elasticity.....	132
5.2.1 Eshelby's inclusion: Stress and strain.....	132
5.3 Concluding comments .....	165
Chapter 6 Eshelby's inclusion with crack.....	166
6.1 Introduction.....	166
6.2 Stresses and displacements on the crack.....	168

6.3 The stresses and the displacements generated by a Westergaard function, $Z$ , for mode I problems.....	172
6.4 Concluding comments .....	180
Chapter 7 Summation of concluding comments .....	181
7.1 Conclusions of thesis.....	181
7.2 Limitations.....	181
7.3 Validation.....	182
7.4 Suggestions for future work.....	182
Reference .....	184



## List of Figures

Figure 1.1: A homogeneous linear elastic solid with volume $V$ and surface $S$ . A subvolume $V_0$ with surface $S_0$ undergoes a permanent deformation. An inclusion is the material inside $V_0$ and the surrounding material is called the matrix. ....	14
Figure 2.1: An elastic body $V$ under applied loads.....	17
Figure 2.2: illustrating the configuration of a undeformed and deformed body.....	18
Figure 2.3: A point force $F$ acting at $x'$ inside an infinite elastic. $V$ is a finite volume bounded by a surface $S$ with an outward normal $\underline{n}$ .....	35
Figure 2.4 Crack opening modes.....	68
Figure 2.5 A schematic illustration of the different fatigue crack closure mechanisms. (a) Plasticity-induced crack closure, (b) oxide-induced crack closure, (c) roughness-induced crack closure, (d) viscous fluid-induced crack closure and (e) phase transformation-induced crack closure .....	73
Figure 3.1 Typical 'compliance' curve of displacement in the neighbourhood of a fatigue crack with load .....	81
Figure 3.2 The variation measuring of the stress intensity factors during the two load cycles, (reproduced from [35]).....	82
Figure 3.3. Schematic idealization of the forces acting at the interface of the plastic enclave and the surrounding elastic material in the CJP model.....	84
Figure 3.4 Comparison between the experimental and CJP predictions of plastic zone area, using the von Mises yield criterion .....	85
Figure 3.5 Illustrating the similitude between the transformations toughening (a) and the plastic zone toughening (b) (reproduced from [38]).....	88
Figure 4.1 A crack in an infinite plate subjected to compressive forces.....	110

Figure 4.2 A new Westergaard's function to model the shear stresses along the crack flanks.....	121
Figure 5.1 The sequence of steps of Eshelby's inclusion problem .....	133
Figure 6.1 Eshelby's inclusion without a crack .....	166
Figure 6.2 The stresses induced from the forces without an Eshelby's inclusion....	167
Figure 6.3 the zero stress on crack with Eshelby inclusion .....	167
Figure 6.4 The x-displacement of subtract Eshelby inclusion from displacements generated by a Westergaard function, $Z$ , with Lamé constants for aluminum ( $\lambda^* = 53.5$ GPa, $\mu = 26.6$ GPa) and $R = 0.5$ mm, $\sigma_{11}^* = -30$ MPa, $\sigma_{12}^* = 0$ and $\sigma_{22}^* = 100$ MPa. ....	178
Figure 6.5 The y-displacement of subtract Eshelby inclusion from displacements generated by a Westergaard function, $Z$ , with Lamé constants for aluminium ( $\lambda^* = 53.5$ GPa, $\mu = 26.6$ GPa) and $R = 0.5$ mm, $\sigma_{11}^* = -30$ MPa, $\sigma_{12}^* = 0$ and $\sigma_{22}^* = 100$ MPa. ....	178
Figure 6.6 As in Figure 6.4, but with inclusion and crack highlighted.....	179
Figure 6.7 As in Figure 6.5, but with inclusion and crack highlighted .....	179

## Nomenclature

$a$  : Crack length

$i$  : Square root of -1

$z$  : Complex coordinate of the points around the crack tip,  $z=x+iy$

$E$  : Young's modulus

$A, B, C, D, F$  : Coefficients

$\sigma_{11}, \sigma_{22}, \sigma_{12}$  : Components of the crack tip stress field

$\sigma_{11}^*, \sigma_{22}^*, \sigma_{12}^*$  : Components of the eigenstress in the inclusion

$\lambda, \mu$  are independent elastic constants, called the Lamé constants

$\nu$  : Poisson's ratio

$K_I$  : Mode I stress intensity factor

$K_{II}$  : Mode II stress intensity factor

$\phi(z), \psi(z)$  : Muskhelishvili complex potentials

$\phi^*(z), \psi^*(z)$  : Muskhelishvili complex potentials arising from eigenstress in inclusion

$r, \theta$  : Notation for polar coordinates

$\sigma_r, \sigma_\theta, \sigma_{r\theta}$  : Components of stress tensor in polar coordinates

$u_1, u_2$  : Horizontal and vertical displacements

$Z$  : Westergaard function for Mode I problems

$T, -\sigma_{0x_1}$  : T- stress

$P, Q$  : Forces

$t$  : The distance from point force to crack tip

$A_1, B_1, C_1$  : Coefficients of the displacement inside the inclusion

$w$  : Complex representation of the displacement relative to a point force

## Acknowledgements

I would like to start these acknowledgements by thanking my supervisor Colin Christopher for the continuous support of my research, for his patience, motivation, and enthusiasm.

My thanks also goes to Neil James and all the staff members of the school of Computing and Mathematics for all the help they provided during my study.

I would also like to extend these acknowledgements to the Libyan government for giving me the opportunity to engage in higher level study and research.

I would like to thank my family: my parents Latafia and Abd allah, and all my sisters and brothers for supporting and prayer for me all the time.

My thanks and appreciation to my beloved husband Dr. Mustafa Taghdi for his support, encouragement and patience throughout my studies.

Thank you my sweetheart and my daughter Sarah for sharing my study journey.

## AUTHOR'S DECLARATION

At no time during the registration for the degree of Doctor of Philosophy has the author been registered for any other University award without prior agreement of the Doctoral College Quality Sub-Committee.

Work submitted for this research degree at the University of Plymouth has not formed part of any other degree either at the University of Plymouth or at another establishment.

This study was financed with the aid of a studentship from the Libyan government.

Word count of main body of thesis: 11,300

Signed

Date 31<sup>st</sup> July 2018

## Chapter 1 Introduction

Fracture mechanics is the branch of applied mechanics that deals with the mechanics of cracked bodies under load and provides tools to predict crack extension and fracture, and hence to assess residual life. Fracture mechanics allows engineers to design fracture-safe and fatigue-reliable structures with defined fatigue life. Some of the mechanisms associated with plastic deformation and crack growth and that influence the magnitude of the range of stress intensity factor at a crack tip are still incompletely understood, among of these are the plasticity-induced crack closure (or crack tip shielding) effect [1, 59].

The term crack closure describes the phenomena of a decrease in fatigue crack growth rate by an apparent decrease in the effective stress intensity factor range,  $\Delta K_{\text{eff}}$ , due to the contact between the crack faces. The lack of understanding comes from the difficulties of measuring this phenomenon and evaluating its impact on the crack driving force [2].

The material inside a plastically transformed region in an isotropic elastic solid can be considered as an “inclusion” in the elastic material with the surrounding material called the “matrix” as shown in Fig 1.1. The solution of the problem of finding the elastic field both in the inclusion and in the surrounding matrix has been given by Eshelby (1957). He solved this problem by imagining cutting around the area which has been deformed and removing it from the matrix [3]. Eshelby showed that the mathematical solution of

this problem can be based on the superposition principle of linear elasticity via a Green's function [4].

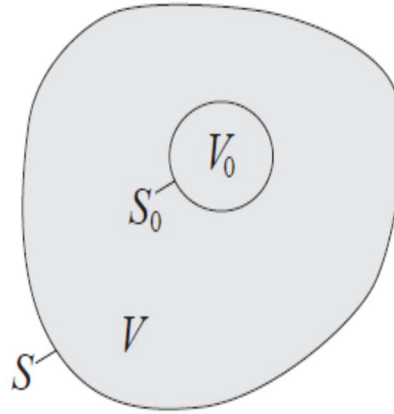


Figure 1.1: A homogeneous linear elastic solid with volume  $V$  and surface  $S$ . A subvolume  $V_0$  with surface  $S_0$  undergoes a permanent deformation. An inclusion is the material inside  $V_0$  and the surrounding material is called the matrix [4].

## 1.1 Motivations and aims of the research

The main motivation for this research topic is contribute to a better understanding of the fatigue crack closure phenomenon and thus help to clarify the nature of the plastic region surrounding the crack tip. This should result in giving better predictions of crack behaviour, for example growth rates. The aim of this thesis is to model the plastic region near the crack tip by using a suitable adaptation of Eshelby's approach. To do this, we translate Eshelby's solution in terms of Muskhelishvili's complex potential functions for 2D elasticity. Using an adaptation of Eshelby's analysis, we find the solution for the stress, strain and displacement fields both inside and outside the inclusion, for the case where the inclusion is a circular disc just in front of the crack tip.

## 1.2 Description of the contents

The current research is intended to contribute to the understanding of how we can use theoretical methods to find stress and displacement around the crack via a modified Eshelby's inclusion in 2D. This work has been organized into seven chapters as follows:

**Chapter 2** gives some background into the most commonly used mathematical models characterising the stress field or the displacement field around the crack tip. After to this, the phenomenon of fatigue crack closure is described.

**Chapter 3** is a literature review of work related to this thesis.

**Chapter 4** presents the complex potential function method and how it can be used to analyse the stresses and displacements around a crack tip.

**Chapter 5** Eshelby's approach to modelling an inclusion is reviewed.

**Chapter 6** the model of the plastic region near the crack tip is analysed and solved to calculate the stress and the displacement around crack tip using Eshelby's approach.

**Chapter 7** provides a summary of this thesis, and gives some recommendations for future work.



## Chapter 2 Background

### 2.1 Introduction

The purpose of this chapter is to describe briefly some of the fundamental concepts of continuum mechanics and fracture mechanics, and classic approaches to modelling cracks and plastic zones that will be employed later in this thesis.

### 2.2 Fundamental theory of continuum mechanics

#### 2.2.1 Stress and equilibrium

Consider a body at equilibrium with a volume,  $V$ , enclosed by a surface,  $S$ . There are two types of forces acting on this body: traction and body forces. The traction,  $T_j$ , acts over the surface area with normal vector,  $n=n_i e_i$ , and  $i=1,2,3$ , and is related to the stresses by [4]

$$\sigma_{ij}n_i=T_j, \quad i,j=1,2,3, \quad (2.1)$$

where it is understood that the summation convention applies and the stress tensor, denoted by  $\sigma_{ij}$ , gives the force per unit area on the  $i$ -face in the  $j$ -direction [4].

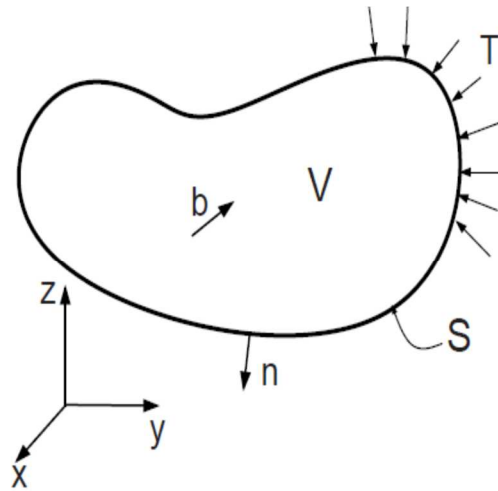


Figure 2.1: An elastic body  $V$  under applied loads [4]

The body force per unit volume,  $b_j$ , as shown in Fig 2.1 represents the external force field. The equilibrium of the forces in the  $j$ -direction can be written as

$$\int_V b_j dV + \int_S T_j dS = 0. \quad (2.2)$$

Substituting from (2.1), we get

$$\int_V b_j dV + \int_S \sigma_{ij} n_i dS = 0. \quad (2.3)$$

Using Gauss's theorem we have,

$$\int_V (b_j + \sigma_{ij,i}) dV = 0,$$

which, since  $V$  is arbitrary, gives the equation for equilibrium as

$$b_j + \sigma_{ij,i} = 0. \quad (2.4)$$

### 2.2.2 Strain and compatibility

Let  $X$  be the coordinate of a point in the undeformed body and  $x$  be the point after deformation as shown in Fig.2.2. The displacement of the point  $X$ , denoted  $u(X)$ , is

$$u(X) = x - X.$$

We can write

$$x = u + X.$$

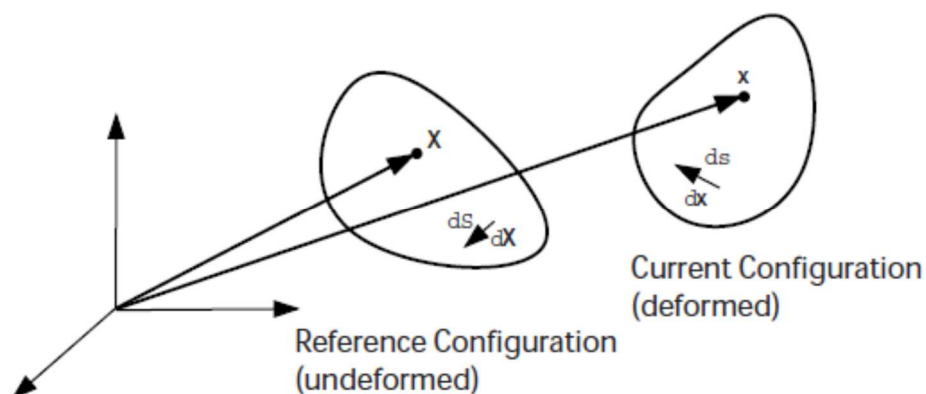


Figure 2.2: illustrating the configuration of a undeformed and deformed body [4]

Now, if  $dS = \sqrt{dX_i dX_i}$  is the length of a small vector  $dX$  in the undeformed body and

$ds = \sqrt{dx_i dx_i}$  is the length of a small vector  $dx$  after deformation, then the strain

tensor,  $\varepsilon_{ij}$ , which is a measure of the body's stretching, can be defined by

$$(ds)^2 - (dS)^2 = (ds + dS)(ds - dS) = 2\varepsilon_{ij} dx_i dx_j.$$

Writing  $ds$  and  $dS$  in terms of the displacements,  $u_i$ , we get,

$$\begin{aligned} (ds)^2 - (dS)^2 &= x_i dx_j - dX_i dX_j = dx_i dx_j \delta_{ij} - \left( \delta_{ij} - \frac{\partial u_i}{\partial x_j} \right) dx_j \left( \delta_{ik} - \frac{\partial u_i}{\partial x_k} \right) dx_k \\ &= dx_i dx_j \delta_{ij} - \left( \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_k} - \frac{\partial u_k}{\partial x_j} - \frac{\partial u_j}{\partial x_k} + \delta_{jk} \right) (u_{j,k} + u_{k,j} - u_{i,j} u_{i,k}) dx_j dx_k \\ &= \varepsilon_{ij} dx_i dx_j + O(x^2), \end{aligned}$$

which, assuming the strain tensor is symmetric, i.e.  $\varepsilon_{ij} = \varepsilon_{ji}$ , gives the strain tensor,  $\varepsilon_{ij}$ , as

$$\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}), \quad (2.5)$$

where we have ignored the terms in  $O(u^2)$ . Now, from the definition of the strain tensor, (2.5), we obtain the following equations

$$\varepsilon_{xx} = u_{x,x}, \quad \varepsilon_{yy} = u_{y,y}, \quad \varepsilon_{zz} = u_{z,z}, \quad \varepsilon_{xy} = \frac{1}{2} (u_{x,y} + u_{y,x}), \quad \varepsilon_{xz} = \frac{1}{2} (u_{x,z} + u_{z,x}), \quad \varepsilon_{yz} = \frac{1}{2} (u_{y,z} + u_{z,y}),$$

hence, we obtain

$$\varepsilon_{xx,yy} = u_{x,xyy}, \quad \varepsilon_{yy,xx} = u_{y,yxx}, \quad \varepsilon_{xy,xy} = \frac{1}{2}(u_{x,xyy} + u_{y,yxx}),$$

and thus  $\varepsilon_{xx,yy}$ ,  $\varepsilon_{yy,xx}$  and  $\varepsilon_{xy,xy}$  must satisfy

$$\varepsilon_{xx,yy} + \varepsilon_{yy,xx} - 2\varepsilon_{xy,xy} = 0.$$

Similarly, we have

$$\varepsilon_{xx,yz} = u_{x,xyz}, \quad \varepsilon_{xy,xz} = \frac{1}{2}(u_{x,xyz} + u_{y,xxz}), \quad \varepsilon_{xz,xy} = \frac{1}{2}(u_{x,xyz} + u_{z,xxz}), \quad \varepsilon_{yz,xx} = \frac{1}{2}(u_{y,xxz} + u_{z,xxz}),$$

which gives

$$\varepsilon_{xx,yz} + \varepsilon_{yy,xz} - 2\varepsilon_{xy,xy} = 0.$$

In the same way we can get two more equations. All six equations can be expressed in index notation by

$$\varepsilon_{pmk} \varepsilon_{qnj} \varepsilon_{jk,nm} = 0, \quad (2.6)$$

where

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{for even permutations of } ijk \\ -1 & \text{for odd permutations of } ijk \\ 0 & \text{for repeated indices} \end{cases}.$$

Conversely, if equation (2.6) holds, one can prove that there exist  $u_i$  such that

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}).$$

### 2.2.3 The equilibrium equation written in terms of displacements

The constitutive equation for the strain as a function of the stress is

$$\sigma = C : \varepsilon \quad \Leftrightarrow \quad \sigma_{ij} = C_{ijkl} \varepsilon_{kl}, \quad (2.7)$$

where  $C$  is a fourth order tensor called the elastic stiffness tensor (or elastic constant tensor). This relation is known as the generalized Hooke's law.

It will become that all isotropic fourth order tensors can be written in the form

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \quad (2.8)$$

where  $\lambda$  and  $\mu$  are independent elastic constants, called the Lamé constants, and

$\delta_{ij}$  is the Kronecker delta defined as

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j. \end{cases}$$

The parameter  $\mu$  is called the shear modulus, and  $\lambda$  is related to Poisson's ratio,  $\nu$ ,

by  $\lambda = \frac{2\mu\nu}{1-2\nu}$ . Substituting (2.8) into (2.7), we obtain

$$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij}. \quad (2.9)$$

In matrix form, Hooke's law for isotropic materials can be written in terms of the Lamé constants as

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ 2\sigma_{23} \\ 2\sigma_{13} \\ 2\sigma_{12} \end{bmatrix} = \begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix}.$$

The equation (2.9) can be rewritten alternatively in terms of  $E$  and  $\nu$  as

$$\sigma_{ij} = \frac{E}{1+\nu} \varepsilon_{ij} + \frac{\nu E}{(1+\nu)(1-2\nu)} \varepsilon_{kk} \delta_{ij},$$

and in matrix form as

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ 2\sigma_{23} \\ 2\sigma_{13} \\ 2\sigma_{12} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-2\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix}.$$

We can invert (2.7) to get a constitutive equation for the stress as function of the strain:

$$\varepsilon = S : \sigma \Leftrightarrow \varepsilon_{ij} = S_{ijkl} \sigma_{kl}, \quad (2.10)$$

where  $S$  is a fourth order tensor called the compliance tensor. We can determine the compliance tensor,  $S_{ijkl}$ , as the inverse of  $C_{ijkl}$ , i.e.,

$$C_{ijkl} S_{klmn} = \frac{1}{2} (\delta_{im} \delta_{jn} + \delta_{in} \delta_{jm})$$

Now, seeking a solution in the form

$$S_{klmn} = \alpha \delta_{kl} \delta_{mn} + \beta (\delta_{km} \delta_{ln} + \delta_{kn} \delta_{lm}),$$

we need

$$\begin{aligned} C_{ijkl} S_{klmn} &= [\lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})] [\alpha \delta_{kl} \delta_{mn} + \beta (\delta_{km} \delta_{ln} + \delta_{kn} \delta_{lm})] \\ &= (3\lambda \alpha + 2\mu \alpha + 2\beta \lambda) \delta_{ij} \delta_{mn} + 2\beta \mu (\delta_{im} \delta_{jn} + \delta_{in} \delta_{jm}) \\ &= \frac{1}{2} (\delta_{im} \delta_{jn} + \delta_{in} \delta_{jm}), \end{aligned}$$

and therefore,



$$\beta = \frac{1}{4\mu}, \alpha = \frac{-\lambda}{2\mu(3\lambda+2\mu)}.$$

It follows that the compliance tensor,  $S_{ijkl}$ , is

$$S_{ijkl} = \frac{-\lambda}{2\mu(3\lambda+2\mu)} \delta_{ij} \delta_{kl} + \frac{1}{4\mu} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}).$$

One can write equation (2.10) as

$$\varepsilon_{ij} = \frac{1}{2\mu} \sigma_{ij} - \frac{\lambda}{2\mu(3\lambda+2\mu)} \sigma_{kk} \delta_{ij}.$$

In matrix form, a constitutive equation for the stress as a function of the strain can be written as

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} \frac{\lambda+\mu}{\mu(3\lambda+2\mu)} & \frac{-\lambda}{2\mu(3\lambda+2\mu)} & \frac{-\lambda}{2\mu(3\lambda+2\mu)} & 0 & 0 & 0 \\ \frac{-\lambda}{2\mu(3\lambda+2\mu)} & \frac{\lambda+\mu}{\mu(3\lambda+2\mu)} & \frac{-\lambda}{2\mu(3\lambda+2\mu)} & 0 & 0 & 0 \\ \frac{-\lambda}{2\mu(3\lambda+2\mu)} & \frac{-\lambda}{2\mu(3\lambda+2\mu)} & \frac{\lambda+\mu}{\mu(3\lambda+2\mu)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\mu} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\mu} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\mu} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ 2\sigma_{23} \\ 2\sigma_{13} \\ 2\sigma_{12} \end{bmatrix}.$$

The components of strain as a function of the components of stress alternatively in terms of  $E$  and  $\nu$  are

$$\varepsilon_{ij} = \frac{1}{E} \left[ (1+\nu) \sigma_{ij} - \nu \sigma_{kk} \delta_{ij} \right],$$

and in matrix form as

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & \frac{-\nu}{E} & \frac{-\nu}{E} & 0 & 0 & 0 \\ \frac{-\nu}{E} & \frac{1}{E} & \frac{-\nu}{E} & 0 & 0 & 0 \\ \frac{-\nu}{E} & \frac{-\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(1+\nu)}{E} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(1+\nu)}{E} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(1+\nu)}{E} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ 2\sigma_{23} \\ 2\sigma_{13} \\ 2\sigma_{12} \end{bmatrix},$$

in which  $E$  and  $\nu$  are given in terms of the Lamé constants by

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}, \nu = \frac{\lambda}{2(\lambda+\mu)}.$$

Now, substituting (2.7) in (2.4) we get,

$$C_{ijkl} \varepsilon_{kl,i} + b_j = 0. \quad (2.11)$$

From (2.11), and the definition of the strain tensor, we obtain

$$\frac{1}{2} C_{ijkl} (u_{k,li} + u_{l,ki}) + b_j = 0.$$

Since  $C_{ijkl} = C_{jikl} = C_{ijlk}$ , then

$$\frac{1}{2} (C_{ijkl} u_{k,li} + C_{ijlk} u_{l,ki}) + b_j = 0.$$

Now, we can combined the equation above into

$$C_{ijkl} u_{k,li} + b_j = 0. \quad (2.12)$$

The following relations hold among Young's modulus  $E$ , Poisson's ratio  $\nu$ , the bulk modulus  $K$ , and the shear modulus  $\mu$ :

$$\lambda = \frac{2\mu\nu}{(1-2\nu)}, \quad E = 2(1+\nu)\mu, \quad \nu = \frac{\lambda}{2(\lambda+\mu)} = \frac{(3K-2\mu)}{2(3K+\mu)}.$$

## 2.3 Linear elastic in 2D

### 2.3.1 Review of plane stress and plane strain field equations

As in [48], plane strain and plane stress are defined as follows.

Plane strain is defined to be a state where

$$u_\alpha = u_\alpha(X_1, X_2), \quad u_3 = 0, \quad \alpha = 1, 2.$$

Plane strain is a good choice when the material is thick.

In a state of plane strain the scalar components of strain  $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$  must be in

the form  $\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0$ ,  $\varepsilon_{\alpha\beta} = \varepsilon_{\alpha\beta}(X_1, X_2)$ , and  $\varepsilon_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha})$ ,  $\alpha, \beta = 1, 2$ .

The scalar components of stress  $\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}$ , must correspondingly have the

form  $\sigma_{13} = \sigma_{23} = 0$ ,  $\sigma_{\alpha\beta} = \sigma_{\alpha\beta}(X_1, X_2)$  and  $\sigma_{33} = \lambda\varepsilon_{\gamma\gamma}$ , where  $\varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} = \varepsilon_{\gamma\gamma}$ . Then stresses in plane strain are

$$\sigma_{\alpha\beta} = 2\mu\varepsilon_{\alpha\beta} + \lambda\varepsilon_{\gamma\gamma}\delta_{\alpha\beta}, \quad (2.13)$$

alternatively, putting this in terms of  $E$  and  $\nu$ , we get

$$\sigma_{\alpha\beta} = \frac{E}{1+\nu}\varepsilon_{\alpha\beta} + \frac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_{\gamma\gamma}\delta_{\alpha\beta}, \quad (2.14)$$

or

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ 2\sigma_{12} \end{bmatrix} = \begin{bmatrix} \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} & \frac{\nu E}{(1+\nu)(1-2\nu)} & 0 \\ \frac{\nu E}{(1+\nu)(1-2\nu)} & \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} & 0 \\ 0 & 0 & \frac{E}{(1+\nu)} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix},$$

where

$$c_{ijkl} = \begin{bmatrix} \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} & \frac{\nu E}{(1+\nu)(1-2\nu)} & 0 \\ \frac{\nu E}{(1+\nu)(1-2\nu)} & \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} & 0 \\ 0 & 0 & \frac{E}{(1+\nu)} \end{bmatrix}.$$

Noting that  $\delta_{\alpha\alpha} = \delta_{11} + \delta_{22} = 2$ , it follows from (2.13) and  $\sigma_{33} = \lambda \varepsilon_{\gamma\gamma}$  that

$$\sigma_{\alpha\alpha} = 2(\mu + \lambda)\varepsilon_{\alpha\alpha} = \frac{2(\mu + \lambda)}{\lambda}\sigma_{33} = \frac{1}{\nu}\sigma_{33}.$$

Since  $\sigma_{kk} = \sigma_{\gamma\gamma} + \sigma_{33} = (1 + \nu)\sigma_{\gamma\gamma}$ , one can obtain the strains in plane strain from the expression

$$\varepsilon_{ij} = \frac{1}{E} \left[ (1 + \nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij} \right].$$

Therefore, the strains in plane strain are

$$\varepsilon_{\alpha\beta} = \frac{(1+\nu)}{E} [\sigma_{\alpha\beta} - \nu \sigma_{\gamma\gamma} \delta_{\alpha\beta}]. \quad (2.15)$$

This means that  $\varepsilon_{ij} = \hat{s}_{ijkl} \sigma_{kl}$ ,

or

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} \frac{1-\nu^2}{E} & \frac{-\nu(1+\nu)}{E} & 0 \\ \frac{-\nu(1+\nu)}{E} & \frac{1-\nu^2}{E} & 0 \\ 0 & 0 & \frac{1+\nu}{E} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ 2\sigma_{12} \end{bmatrix},$$

where

$$\hat{s}_{ijkl} = \begin{bmatrix} \frac{1-\nu^2}{E} & \frac{-\nu(1+\nu)}{E} & 0 \\ \frac{-\nu(1+\nu)}{E} & \frac{1-\nu^2}{E} & 0 \\ 0 & 0 & \frac{1+\nu}{E} \end{bmatrix}.$$

By calculations,  $[\hat{s}]^{-1} = c$ .

Let  $C_{ijkl}$  be the elastic stiffness tensor of a homogeneous solid and  $S_{ijkl}$  be the compliance tensor ( $S_{ijkl}$  is the inverse of  $C_{ijkl}$ ). Let  $c_{ijkl}$  be the 2D elastic stiffness tensor.

Therefore,

$$\sigma_{ij} = c_{ijkl} \varepsilon_{kl}, \text{ where } i, j, k, l = 1, 2. \quad (\text{plain strain})$$

Where  $C_{ijkl} = c_{ijkl}$  for  $i, j, k, l = 1, 2$ .

**Plane stress** is defined as a state where

$$\sigma_{\alpha\beta} = \sigma_{\alpha\beta}(X_1, X_2), \sigma_{13} = \sigma_{23} = \sigma_{33} = 0, \alpha, \beta = 1, 2.$$

Plane stress is a good choice when the material is thin. Since  $\sigma_{kk} = \sigma_{11} + \sigma_{22} = \sigma_{\gamma\gamma}$ , one can get the scalar components of strain in plane stress from the expression

$$\varepsilon_{ij} = \frac{1}{E} \left[ (1+\nu) \sigma_{ij} - \nu \sigma_{kk} \delta_{ij} \right],$$

where  $\varepsilon_{13} = \varepsilon_{23} = 0$ ,  $\varepsilon_{\alpha\beta} = \varepsilon_{\alpha\beta}(X_1, X_2)$ , and  $\varepsilon_{33} = \frac{-\nu}{E} \sigma_{\gamma\gamma}$  by

$$\varepsilon_{\alpha\beta} = \frac{1}{E} \left[ (1+\nu) \sigma_{\alpha\beta} - \nu \sigma_{\gamma\gamma} \delta_{\alpha\beta} \right], \quad (2.16)$$

the strain in plane stress is given by

$$\varepsilon_{ij} = \tilde{s}_{ijkl} \sigma_{kl}, i, j, k, l = 1, 2.$$

Or

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & \frac{-\nu}{E} & 0 \\ \frac{-\nu}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{1+\nu}{E} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ 2\sigma_{12} \end{bmatrix},$$

where

$$\tilde{s}_{ijkl} = \begin{bmatrix} \frac{1}{E} & \frac{-\nu}{E} & 0 \\ \frac{-\nu}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{1+\nu}{E} \end{bmatrix}.$$

Obviously,

$$[\tilde{s}]^{-1} = \tilde{c} = \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0 \\ \frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & \frac{E}{1+\nu} \end{bmatrix}.$$



Since  $\varepsilon_{\alpha\alpha} = \frac{1}{E}[(1+\nu)\sigma_{\alpha\alpha} - \nu\sigma_{\gamma\gamma}\delta_{\alpha\alpha}] = \frac{1-\nu}{E}\sigma_{\alpha\alpha} = -\frac{1-\nu}{\nu}\varepsilon_{33}$ , then the component of strain  $\varepsilon_{33}$  in terms of  $\lambda$  and  $\mu$  is

$$\varepsilon_{33} = \frac{-\nu}{1-\nu}\varepsilon_{\alpha\alpha} = \frac{-\lambda}{\lambda+2\mu}\varepsilon_{\alpha\alpha}.$$

Now, from  $\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}$  and by noting that  $\varepsilon_{kk} = \varepsilon_{\gamma\gamma} + \varepsilon_{33} = \frac{2\mu}{\lambda+2\mu}\varepsilon_{\gamma\gamma}$ , the components of stress in plane stress are given by

$$\sigma_{\alpha\beta} = 2\mu\varepsilon_{\alpha\beta} + \frac{2\mu\lambda}{\lambda+2\mu}\varepsilon_{\gamma\gamma}\delta_{\alpha\beta}, \quad (2.17)$$

or, alternatively in terms of  $E$  and  $\nu$ ,

$$\sigma_{\alpha\beta} = \frac{E}{1+\nu}\varepsilon_{\alpha\beta} + \frac{\nu E}{1-\nu^2}\varepsilon_{\gamma\gamma}\delta_{\alpha\beta}, \quad (2.18)$$

the stress in plane stress is therefore related to the strain by

$$\sigma_{ij} = \tilde{c}_{ijkl}\varepsilon_{kl}, \quad i, j, k, l = 1, 2.$$

or

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ 2\sigma_{12} \end{bmatrix} = \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0 \\ \frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & \frac{E}{1+\nu} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \end{bmatrix},$$

where

$$\tilde{c}_{ijkl} = \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0 \\ \frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & \frac{E}{1+\nu} \end{bmatrix}.$$

By comparing the equations of plane strain and plane stress elasticity (2.14), (2.15), (2.16) and (2.17), one can see that each of plane strain equations can be transformed into its corresponding plane stress equation, and vice versa, by for example, to go from

plane stress to plane strain,  $E \rightarrow \frac{E}{1-\nu^2}$ ,  $\nu \rightarrow \frac{\nu}{1-\nu}$ ,  $\lambda^* \rightarrow \frac{2\mu\lambda}{2\mu-\lambda}$  and  $\mu \rightarrow \mu$ .

## 2.4 The linear elastic field by using the Green's function

### 2.4.1 Equilibrium equation for an infinite body

Let  $F = F_j \delta(\mathbf{x} - \mathbf{x}') \mathbf{e}_j$  be a constant point force acting at  $\mathbf{x}'$  in an infinite body. Suppose that  $V$  is an arbitrary volume bounded by a surface  $S$  with an outward normal  $\underline{n}$  as shown in Fig.1.3. Then, the displacement field generated by this applied force is given by

$$u_i(x) = G_{ij}(x-x') F_j,$$

for some Green's function  $G_{ij}(x, x') = G_{ij}(x-x')$  which only depends on the displacement between the points. Now,

$$u_{i,m}(x) = G_{ij,m}(x-x') F_j,$$

and, by using Hooke's law, the stress field can be written as

$$\sigma_{kp}(x) = C_{kpim} G_{ij,m}(x-x') F_j. \quad (2.19)$$

Since  $V$  is the volume surrounding the point  $x_0$ , then the force  $F$  must be balanced by the tractions acting over the surface  $S$ . This means that

$$F_k + \int_S \sigma_{kp}(X) n_p(X) dS(X) = 0,$$

rewriting this equation using (2.19), we get

$$F_k + \int_S C_{kpim} G_{ij,m}(x-x') n_p(X) F_j dS(X) = 0.$$

Now, by using Gauss's theorem on the surface integral, we have

$$F_k + \int_V C_{kpim} G_{ij,m}(x-x') F_j dV(X) = 0. \quad (2.20)$$

Since the three dimensional Dirac delta function is defined as

$$\int_V \delta(\mathbf{x}-\mathbf{x}') dV(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x}' \in V \\ 0 & \text{if } \mathbf{x}' \notin V, \end{cases}$$

then we can write (2.20) in the form

$$\int_V [C_{k\text{pim}} G_{ij,\text{mp}}(\mathbf{x}-\mathbf{x}') + \delta_{kj} \delta(\mathbf{x}-\mathbf{x}')] F_j dV(\mathbf{x}) = 0.$$

Consequently, since  $V$  is arbitrary,

$$C_{k\text{pim}} G_{ij,\text{mp}}(\mathbf{x}-\mathbf{x}') + \delta_{kj} \delta(\mathbf{x}-\mathbf{x}') = 0.$$

This is equilibrium equation for an infinite body.

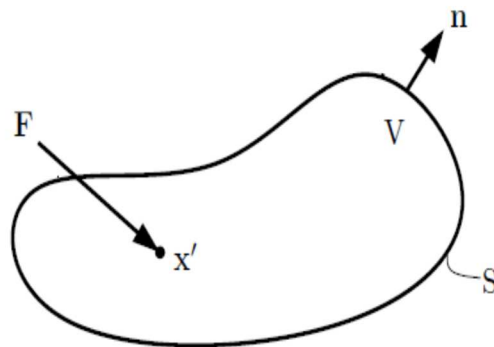


Figure 2.3: A point force  $F$  acting at  $\mathbf{x}'$  inside an infinite elastic.  $V$  is a finite volume bounded by a surface  $S$  with an outward normal  $\underline{n}$  [4].

### 2.4.2 Green's function in Fourier space

The Fourier transform of the elastic Green's function is defined as

$$g_{ij}(k) = \int_{R^3} \exp(ik \cdot x) G_{ij}(x) dx.$$

Standard theory shows that we can invert this expression to obtain  $G_{ij}$  in terms of  $g_{ij}$  :

$$G_{ij}(x) = \frac{1}{(2\pi)^3} \int_{R^3} \exp(-ik \cdot x) g_{ij}(k) dk.$$

Using the formula above we can express the three dimensional Dirac delta function as

$$\delta(x) = \frac{1}{(2\pi)^3} \int_{R^3} \exp(-ik \cdot x) dk.$$

The equilibrium equation for the elastic Green's function can now be solved in the Fourier space. Substituting in the definitions of  $G_{ij}(x)$  and  $\delta(x)$  (setting  $x'=0$ ), we have

$$\frac{1}{(2\pi)^3} \int_{R^3} [C_{kpim} g_{ij}(k) \frac{\partial^2}{\partial x_m \partial x_p} + \delta_{kj}] \exp(-ik \cdot x) dk = 0.$$

The vector  $z$  is defined as

$$z = \frac{k}{|k|}.$$

Therefore,

$$\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} [-C_{kpim} z_m z_p k^2 g_{ij}(k) + \delta_{kj}] \exp(-ik \cdot x) dk = 0.$$

It follows that,

$$C_{kpim} z_m z_p k^2 g_{ij}(k) = \delta_{jk}. \quad (2.21)$$

The tensor  $(zz)_{ki}$  is defined by

$$(zz)_{ki} \equiv C_{pkim} z_p z_m.$$

Using this definition by substituting in equation (2.21), we get

$$(zz)_{ki} g_{ij} k^2 = \delta_{kj}.$$

The inverse of the  $(zz)_{ij}$  tensor is defined such that

$$(zz)_{nk}^{-1} (zz)_{ki} = \delta_{ni}.$$

Hence the Green's function in Fourier space is

$$g_{ij}(k) = \frac{(zz)_{ij}^{-1}}{k^2}.$$

### 2.4.3 The elastic stress field in 2D

The main objective of the current calculation is finding the stress, strain and displacement fields by using the Green's function in 2D. The calculations below may not be new, but are included for lack of a suitable reference. We start first by calculating the Green's function in 2D as following:

The displacement field caused by applied force  $F_\beta$  is

$$u_\alpha(x) = G_{\alpha\beta}(x-x')F_\beta, \quad (2.22)$$

which gives the displacement gradients

$$u_{\alpha,\gamma}(x) = G_{\alpha\beta,\gamma}(x-x')F_\beta. \quad (2.23)$$

From Hooke's law  $\sigma_{\delta\epsilon}(x) = C_{\delta\epsilon\alpha\gamma}\epsilon_{\alpha\gamma}$  and the relation between the strain and the displacement, we have

$$\begin{aligned} \epsilon_{\alpha\gamma} &= \frac{1}{2}(u_{\alpha,\gamma} + u_{\gamma,\alpha}) \\ &= \frac{1}{2}[G_{\alpha\beta,\gamma}(x-x')F_\beta + G_{\gamma\beta,\alpha}(x-x')F_\beta] \\ &= \frac{1}{2}[G_{\alpha\beta,\gamma}(x-x') + G_{\gamma\beta,\alpha}(x-x')]F_\beta, \end{aligned} \quad (2.24)$$

the stress field can be obtained from Hook's law as

$$\begin{aligned}\sigma_{\delta\epsilon}(x) &= C_{\delta\epsilon\alpha\gamma} \epsilon_{\alpha\gamma} \\ &= C_{\delta\epsilon\alpha\gamma} G_{\alpha\beta,\gamma}(x-x') F_{\beta}.\end{aligned}\quad (2.25)$$

As before, since  $V$  is the volume surrounding the point  $x_0$ , then the force  $F$  must be balanced by the tractions acting over the surface  $S$ . This means that

$$F_{\delta} + \int_S \sigma_{\delta\epsilon}(x) n_{\epsilon}(x) ds(x) = 0. \quad (2.26)$$

Rewriting this equation using (2.25), we get

$$F_{\delta} + \int_S [C_{\delta\epsilon\alpha\gamma} G_{\alpha\beta,\gamma}(x-x') F_{\beta} n_{\epsilon}(x) ds(x) = 0. \quad (2.27)$$

Now, using Gauss's theorem on the surface integral, we have

$$F_{\delta} + \int_V C_{\delta\epsilon\alpha\gamma} G_{\alpha\beta,\gamma\epsilon}(x-x') F_{\beta} dV(x) = 0. \quad (2.28)$$

Since the two dimensional Dirac delta function is defined as

$$\int_V \delta(x-x') dV(x) = \begin{cases} 1 & \text{if } x' \in V \\ 0 & \text{if } x' \notin V, \end{cases}$$

then we can write (2.28) in the form



$$\int_V [C_{\delta\epsilon\alpha\gamma} G_{\alpha\beta,\gamma\epsilon}(x-x') F_\beta + F_\delta \delta(x-x')] dV(x) = 0.$$

Factoring  $F_\beta$  after replacing  $F_\delta$  with  $F_\beta \delta_{\beta\delta}$  we obtain

$$\int_V [C_{\delta\epsilon\alpha\gamma} G_{\alpha\beta,\gamma\epsilon}(x-x') + \delta_{\delta\beta} \delta(x-x')] F_\beta dV(x) = 0. \quad (2.29)$$

Consequently, since  $V$  is an arbitrary,

$$C_{\delta\epsilon\alpha\gamma} G_{\alpha\beta,\gamma\epsilon}(x-x') + \delta_{\delta\beta} \delta(x-x') = 0, \quad (2.30)$$

which is the equilibrium equation satisfied by the Green's function in an infinite elastic body. Now, one can solve equation (2.30) using Fourier transforms. The Fourier transform of the elastic Green's function is defined as

$$g_{\alpha\beta}(k) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ik \cdot x} G_{\alpha\beta}(x) dx. \quad (2.31)$$

Standard theory shows that we can invert this expression to obtain  $G_{\alpha\beta}$  in terms of

$g_{\alpha\beta}$  :

$$G_{\alpha\beta}(x) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} e^{-ik \cdot x} g_{\alpha\beta}(k) dk. \quad (2.32)$$

Using the formula above we can express the two dimensional Dirac delta function as

$$\delta(x) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} e^{-ik \cdot x} dk.$$

The equilibrium equation for the elastic Green's function can now be solved in the Fourier space. From (2.30) and using definition of  $G_{\alpha\beta}(x)$ ,  $\delta(x)$ , when  $x'=0$ , we obtain

$$\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \left[ C_{\delta\epsilon\alpha\gamma} \frac{\partial^2}{\partial x_\gamma \partial x_\epsilon} g_{\alpha\beta}(k) + \delta_{\delta\beta} \right] e^{-ik \cdot x} dk = 0. \quad (2.33)$$

We define the vector  $z$  as

$$z = \frac{k}{|k|}. \quad (2.34)$$

Equation (2.33) then simplifies to

$$\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} e^{-ik \cdot x} \left[ -C_{\delta\epsilon\alpha\gamma} z_\gamma z_\epsilon k^2 g_{\alpha\beta}(k) + \delta_{\delta\beta} \right] dk = 0. \quad (2.35)$$

It follows that

$$-C_{\delta\epsilon\alpha\gamma} z_\gamma z_\epsilon k^2 g_{\alpha\beta}(k) + \delta_{\delta\beta} = 0.$$

Therefore,

$$C_{\delta\epsilon\alpha\gamma} z_\gamma z_\epsilon k^2 g_{\alpha\beta}(k) = \delta_{\delta\beta}. \quad (2.36)$$

We define the tensor  $(zz)_{\delta\alpha}$  by

$$(zz)_{\delta\alpha} = C_{\delta\epsilon\alpha\gamma} z_\gamma z_\epsilon. \quad (2.37)$$

Using this definition, and substituting in equation (2.36), we get

$$(zz)_{\delta\alpha} k^2 g_{\alpha\beta}(k) = \delta_{\delta\beta}. \quad (2.38)$$

The inverse of the  $(zz)_{\delta\alpha}$  tensor is defined by

$$(zz)_{\epsilon\delta}^{-1} (zz)_{\delta\alpha} = \delta_{\epsilon\alpha}. \quad (2.39)$$

Therefore

$$\begin{aligned} (zz)_{\epsilon\delta}^{-1} (zz)_{\delta\alpha} k^2 g_{\alpha\beta}(k) &= (zz)_{\epsilon\delta}^{-1} \delta_{\delta\beta} \\ \Rightarrow \delta_{\epsilon\alpha} k^2 g_{\alpha\beta}(k) &= (zz)_{\epsilon\beta}^{-1} \\ \Rightarrow k^2 g_{\epsilon\beta}(k) &= (zz)_{\epsilon\beta}^{-1}. \end{aligned}$$

It follows that the Green's function in Fourier space is

$$g_{\varepsilon\beta}(k) = \frac{(zz)_{\varepsilon\beta}^{-1}}{k^2}. \quad (2.40)$$

Now,

$$\begin{aligned} G_{\alpha\beta}(x) &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} e^{-ik \cdot x} g_{\alpha\beta}(k) dk \\ &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} e^{-ik \cdot x} \frac{(zz)_{\alpha\beta}^{-1}}{k^2} dk. \end{aligned} \quad (2.41)$$

Now we can calculate  $G_{11}(x)$ ,  $G_{22}(x)$  and  $G_{12}(x)$  as follows:

From (2.41) we have

$$G_{11} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{\infty} e^{-ikx \cos(\varphi-\theta)} g_{11}(k) dk d\theta. \quad (2.42)$$

The integral can be written as

$$G_{11} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{\infty} e^{-ikx \cos(\varphi-\theta)} k \frac{(zz)_{11}^{-1}}{k^2} dk d\theta. \quad (2.43)$$

The definition of  $(zz)_{\alpha\beta}$  gives

$$\begin{aligned}
(zz)_{\alpha\beta} &= z_{\delta} z_{\gamma} C_{\alpha\delta\beta\gamma} \\
&= z_{\delta} z_{\gamma} [\lambda^* \delta_{\alpha\delta} \delta_{\beta\gamma} + \mu(\delta_{\alpha\beta} \delta_{\delta\gamma} + \delta_{\alpha\gamma} \delta_{\beta\delta})] \\
&= \lambda^* z_{\alpha} z_{\beta} + \mu(z_{\alpha} z_{\beta} + \delta_{\alpha\beta} z_{\gamma} z_{\gamma}) \\
&= (\lambda^* + \mu) z_{\alpha} z_{\beta} + \mu \delta_{\alpha\beta},
\end{aligned} \tag{2.44}$$

where  $\lambda^* = \frac{3-k}{k-1} \mu$  and

$$k = \begin{cases} 3-4\nu & \text{for plane strain} \\ \frac{3-\nu}{1+\nu} & \text{for plane stress} \end{cases}. \tag{2.45}$$

We define  $(zz)_{\alpha\beta}$  as

$$(zz)_{\alpha\beta} = \mu \left( \delta_{\alpha\beta} + \frac{\lambda^* + \mu}{\mu} z_{\alpha} z_{\beta} \right). \tag{2.46}$$

Therefore the inverse can be written as

$$(zz)_{\alpha\beta}^{-1} = \frac{1}{\mu} \left( \delta_{\alpha\beta} - \frac{\lambda^* + \mu}{\lambda^* + 2\mu} z_{\alpha} z_{\beta} \right). \tag{2.47}$$

Substituting (2.47) into equation (2.43) we obtain

$$G_{11} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{\infty} e^{-ikx \cos(\varphi-\theta)} \frac{1}{k} \left[ \frac{1}{\mu} \left( \delta_{11} - \frac{\lambda^* + \mu}{\lambda^* + 2\mu} z_1 z_1 \right) \right] dk d\theta$$

$$= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^\infty e^{-ikx \cos(\varphi-\theta)} \frac{1}{k\mu} \left[ 1 - \frac{\lambda^* + \mu}{\lambda^* + 2\mu} \cos^2 \theta \right] dk d\theta, \quad (2.48)$$

in which  $z_1 = \cos \theta$ . Thus we can simplify equation (2.48) to

$$G_{11} = \frac{1}{4\pi^2 \mu} \left[ \lim_{R \rightarrow \infty} \left( \int_0^{2\pi} \int_0^R e^{-ikx \cos(\varphi-\theta)} \frac{1}{k} dk d\theta - \frac{\lambda^* + \mu}{\lambda^* + 2\mu} \int_0^{2\pi} \int_0^R e^{-ikx \cos(\varphi-\theta)} \frac{1}{k} \cos^2 \theta dk d\theta \right) \right]. \quad (2.49)$$

We now wish to evaluate this equations. To simplify writing, let  $I$  denote  $G_{11}$ .

Differentiating (2.49) shows that

$$\begin{aligned} \frac{\partial I}{\partial x} = \frac{1}{4\pi^2 \mu} & \left[ \lim_{R \rightarrow \infty} \left( \int_0^{2\pi} \int_0^R e^{-ikx \cos(\varphi-\theta)} \frac{-ik \cos(\varphi-\theta)}{k} dk d\theta \right. \right. \\ & \left. \left. - \frac{\lambda^* + \mu}{\lambda^* + 2\mu} \int_0^{2\pi} \int_0^R e^{-ikx \cos(\varphi-\theta)} \frac{-ik \cos(\varphi-\theta)}{k} \cos^2 \theta dk d\theta \right) \right], \end{aligned} \quad (2.50)$$

which simplifies (2.50) to

$$\begin{aligned} \frac{\partial I}{\partial x} = \frac{-i}{4\pi^2 \mu} & \left[ \lim_{R \rightarrow \infty} \left( \int_0^{2\pi} \int_0^R e^{-ikx \cos(\varphi-\theta)} \cos(\varphi-\theta) dk d\theta \right. \right. \\ & \left. \left. - \frac{\lambda^* + \mu}{\lambda^* + 2\mu} \int_0^{2\pi} \int_0^R e^{-ikx \cos(\varphi-\theta)} \cos(\varphi-\theta) \cos^2 \theta dk d\theta \right) \right]. \end{aligned} \quad (2.51)$$

We evaluate the terms separately. Let

$$\begin{aligned}
I_1 &= \lim_{R \rightarrow \infty} \int_0^{2\pi} \int_0^R e^{-ikx \cos(\varphi-\theta)} \cos(\varphi-\theta) dk d\theta = \lim_{R \rightarrow \infty} \int_0^{2\pi} \left. \frac{e^{-ikx \cos(\varphi-\theta)}}{-ix} \right|_0^R d\theta \\
&= \frac{-1}{ix} \lim_{R \rightarrow \infty} \int_0^{2\pi} [e^{-iRx \cos(\varphi-\theta)} - 1] d\theta.
\end{aligned} \tag{2.52}$$

Thus, letting  $\theta \Rightarrow \theta_1 - \frac{\pi}{2} + \varphi$ ,  $d\theta \Rightarrow d\theta_1$ , then  $\cos(\varphi-\theta) = \cos(\frac{\pi}{2} - \theta_1) = \sin\theta_1$ . So that

substituting this into (2.52) gives

$$I_1 = \frac{-1}{ix} \lim_{R \rightarrow \infty} \int_0^{2\pi} [e^{-iRx \sin\theta_1} - 1] d\theta_1. \tag{2.53}$$

It follows from a result of Legendre that

$$I_1 = \frac{-1}{ix} \lim_{R \rightarrow \infty} [2\pi J_0(Rx) - 2\pi] = \frac{2\pi}{ix}. \tag{2.54}$$

Let

$$I_2 = \lim_{R \rightarrow \infty} \int_0^{2\pi} \int_0^R e^{-ikx \cos(\varphi-\theta)} \cos(\varphi-\theta) \cos^2\theta dk d\theta. \tag{2.55}$$

Integrating (2.55) with respect  $k$  gives

$$\begin{aligned}
I_2 &= \lim_{R \rightarrow \infty} \int_0^{2\pi} \frac{e^{-ikx \cos(\varphi-\theta)}}{-ix \cos(\varphi-\theta)} \cos(\varphi-\theta) \cos^2 \theta \bigg|_0^R d\theta \\
&= \frac{-1}{ix} \lim_{R \rightarrow \infty} \int_0^{2\pi} e^{-ikx \cos(\varphi-\theta)} \cos^2 \theta d\theta \\
&= \frac{-1}{ix} \lim_{R \rightarrow \infty} \int_0^{2\pi} [e^{-iR x \cos(\varphi-\theta)} \cos^2 \theta - \cos^2 \theta] d\theta.
\end{aligned} \tag{2.56}$$

Let  $\theta \Rightarrow (\theta_1 - \frac{\pi}{2} + \varphi)$ . Then  $\cos(\varphi-\theta) = \cos(\frac{\pi}{2} - \theta_1) = \sin \theta_1$ . By substituting this into (2.56), we have

$$I_2 = \frac{-1}{ix} \lim_{R \rightarrow \infty} \left[ \int_0^{2\pi} [e^{-iR x \sin \theta_1} \cos^2(\theta_1 - \frac{\pi}{2} + \varphi) - \cos^2(\theta_1 - \frac{\pi}{2} + \varphi)] d\theta_1 \right]. \tag{2.57}$$

Using trigonometric formula ( $\cos(\varphi-\theta) = \cos(\frac{\pi}{2} - \theta_1) = \sin \theta_1$ ) to simplify (2.57), we get

$$\begin{aligned}
I_2 &= \frac{-1}{ix} \lim_{R \rightarrow \infty} \left[ \int_0^{2\pi} [e^{-iR x \sin \theta_1} \cos^2 \varphi \sin^2 \theta_1 d\theta_1 + 2 \int_0^{2\pi} e^{-iR x \sin \theta_1} \sin \theta_1 \cos \varphi \cos \theta_1 \sin \varphi d\theta_1 \right. \\
&\quad \left. + \int_0^{2\pi} [e^{-iR x \sin \theta_1} \cos^2 \theta_1 \sin^2 \varphi d\theta_1 - \cos^2(\theta_1 - \frac{\pi}{2} + \varphi) d\theta_1] \right].
\end{aligned} \tag{2.58}$$

We can use partial integration to obtain

$$\frac{-1}{ix} \lim_{R \rightarrow \infty} \int_0^{2\pi} e^{-iR x \sin \theta_1} \cos^2 \theta_1 \sin^2 \varphi d\theta_1 = \frac{-\sin^2 \varphi}{ix} \lim_{R \rightarrow \infty} \left[ \int_0^{2\pi} [e^{-iR x \sin \theta_1} \cos^2 \theta_1 d\theta_1] \right]. \tag{2.59}$$



Taking  $u = \cos\theta_1 \Rightarrow du = -\sin\theta_1 d\theta_1$  and  $dv = \cos\theta_1 e^{-iR_x \sin\theta_1} d\theta_1 \Rightarrow v = \frac{e^{-iR_x \sin\theta_1}}{-iR_x}$ , we find

$$\begin{aligned}
 \int_0^{2\pi} e^{-iR_x \sin\theta_1} \cos^2\theta_1 d\theta_1 &= \frac{-e^{-iR_x \sin\theta_1} \cos\theta_1}{iR_x} \Bigg|_0^{2\pi} - \int_0^{2\pi} \left[ \frac{e^{-iR_x \sin\theta_1}}{-iR_x} (-\sin\theta_1 d\theta_1) \right] \\
 &= \left( \frac{-1}{iR_x} + \frac{1}{iR_x} \right) - \int_0^{2\pi} \frac{e^{-iR_x \sin\theta_1}}{iR_x} \sin\theta_1 d\theta_1 \\
 &= \frac{-1}{iR_x} \int_0^{2\pi} e^{-iR_x \sin\theta} \sin\theta d\theta.
 \end{aligned} \tag{2.60}$$

To evaluate this expressions, let

$$I^* = \int_0^{2\pi} e^{-iR_x \sin\theta} d\theta = 2\pi J_0(R_x),$$

then

$$\frac{\partial I^*}{\partial R} = \int_0^{2\pi} e^{-iR_x \sin\theta} (-ix \sin\theta) d\theta,$$

and

$$2\pi x J'_0(R_x) = \int_0^{2\pi} e^{-iR_x \sin\theta} (-ix \sin\theta) d\theta \Rightarrow \frac{-2\pi J'_0(R_x)}{i} = \int_0^{2\pi} e^{-iR_x \sin\theta} \sin\theta d\theta. \tag{2.61}$$

It follows from (2.59) that

$$\begin{aligned}
\frac{-1}{ix} \lim_{R \rightarrow \infty} \int_0^{2\pi} e^{-iR x \sin \theta} \cos^2 \theta \sin^2 \varphi d\theta &= \frac{-\sin^2 \varphi}{ix} \lim_{R \rightarrow \infty} \left[ \frac{-1}{iR x} \int_0^{2\pi} e^{-iR x \sin \theta} \sin \theta d\theta \right] \\
&= \frac{-\sin^2 \varphi}{ix} \lim_{R \rightarrow \infty} \left[ \frac{-1}{iR x} \left( \frac{-2\pi J'_0(Rx)}{i} \right) \right] = 0.
\end{aligned} \tag{2.62}$$

Now, we need to find the integral

$$\frac{-1}{ix} \lim_{R \rightarrow \infty} \left[ \int_0^{2\pi} 2e^{-iR x \sin \theta} \sin \theta \cos \theta \sin \varphi \cos \varphi d\theta \right] = \frac{-2 \sin \varphi \cos \varphi}{ix} \lim_{R \rightarrow \infty} \left[ \int_0^{2\pi} e^{-iR x \sin \theta} \sin \theta \cos \theta d\theta \right]. \tag{2.63}$$

By a similar method we use integration by parts with  $u = \sin \theta$ ,  $du = \cos \theta d\theta$  and

$$dv = \cos \theta e^{-iR x \sin \theta} d\theta, v = \frac{e^{-iR x \sin \theta}}{-iR x} \text{ to give}$$

$$\begin{aligned}
\int_0^{2\pi} e^{-iR x \sin \theta} \sin \theta \cos \theta d\theta &= \left. \frac{-e^{-iR x \sin \theta} \sin \theta}{iR x} \right|_0^{2\pi} - \int_0^{2\pi} \left[ \frac{e^{-iR x \sin \theta}}{-iR x} \cos \theta d\theta \right] \\
&= \int_0^{2\pi} \frac{e^{-iR x \sin \theta}}{iR x} \cos \theta d\theta = \frac{1}{iR x} \left[ \frac{-e^{-iR x \sin \theta}}{iR x} \right]_0^{2\pi} \\
&= \frac{-1}{i^2 R^2 x^2} [e^{-iR x \sin \theta}]_0^{2\pi} = \frac{-1}{i^2 R^2 x^2} (1-1) = 0.
\end{aligned} \tag{2.64}$$

Substituting this into equation (2.63), we obtain

$$\frac{-1}{ix} \lim_{R \rightarrow \infty} \left[ \int_0^{2\pi} 2e^{-iR x \sin \theta} \sin \theta \cos \theta \sin \varphi \cos \varphi d\theta \right] = 0. \tag{2.65}$$

Now we want to find the integral

$$\frac{-1}{ix} \lim_{R \rightarrow \infty} \int_0^{2\pi} e^{-iR x \sin \theta} \cos^2 \varphi \sin^2 \theta d\theta, \quad (2.66)$$

which can be simplified to

$$\begin{aligned} \frac{-1}{ix} \lim_{R \rightarrow \infty} \int_0^{2\pi} e^{-iR x \sin \theta} \cos^2 \varphi \sin^2 \theta d\theta &= \frac{-\cos^2 \varphi}{ix} \lim_{R \rightarrow \infty} \left[ \int_0^{2\pi} [e^{-iR x \sin \theta} \sin^2 \theta d\theta] \right. \\ &= \frac{-\cos^2 \varphi}{ix} \lim_{R \rightarrow \infty} \left[ \int_0^{2\pi} [e^{-iR x \sin \theta} (1 - \cos^2 \theta) d\theta] \right. \\ &= \frac{-\cos^2 \varphi}{ix} \lim_{R \rightarrow \infty} \left[ \int_0^{2\pi} e^{-iR x \sin \theta} d\theta - \int_0^{2\pi} e^{-iR x \sin \theta} \cos^2 \theta d\theta \right] \\ &= \frac{-\cos^2 \varphi}{ix} \lim_{R \rightarrow \infty} \left[ 2\pi J_0(Rx) + \frac{2\pi J'_0(Rx)}{Rx^2} \right] = 0. \end{aligned} \quad (2.67)$$

Now, we can calculate the integral

$$\frac{-1}{ix} \lim_{R \rightarrow \infty} \int_0^{2\pi} \cos^2 \left( \theta - \frac{\pi}{2} + \varphi \right) d\theta, \quad (2.68)$$

which can be simplified to

$$\begin{aligned} \frac{-1}{ix} \lim_{R \rightarrow \infty} \int_0^{2\pi} \cos^2 \left( \theta - \frac{\pi}{2} + \varphi \right) d\theta &= \frac{-1}{ix} \lim_{R \rightarrow \infty} \left[ \int_0^{2\pi} [\cos^2 \varphi \sin^2 \theta d\theta \right. \\ &\quad \left. + 2 \int_0^{2\pi} \sin \theta \cos \varphi \cos \theta \sin \varphi d\theta + \int_0^{2\pi} [\cos^2 \theta \sin^2 \varphi d\theta] \right]. \end{aligned} \quad (2.69)$$

Simplifying and evaluating equation (2.69) leads to

$$\begin{aligned}
 \frac{-1}{ix} \lim_{R \rightarrow \infty} \int_0^{2\pi} \cos^2(\theta - \frac{\pi}{2} + \varphi) d\theta &= \frac{-1}{ix} \lim_{R \rightarrow \infty} [\cos^2 \varphi \int_0^{2\pi} \sin^2 \theta d\theta \\
 &+ 2\sin \varphi \cos \varphi \int_0^{2\pi} \sin \theta \cos \theta d\theta + \sin^2 \varphi \int_0^{2\pi} \cos^2 \theta d\theta] \\
 &= \frac{-1}{ix} \lim_{R \rightarrow \infty} [\pi \cos^2 \varphi + 2\sin \varphi \cos \varphi (0) + \pi \sin^2 \varphi] = \frac{-1}{ix} \lim_{R \rightarrow \infty} (\pi) = \frac{-\pi}{ix}.
 \end{aligned} \tag{2.70}$$

Using the results of equations (2.62), (2.65), (2.67) and (2.70) into equation (2.58), we get

$$I_2 = \frac{-1}{ix} \lim_{R \rightarrow \infty} \left[ \int_0^{2\pi} [e^{-iRx \sin \theta} \cos^2(\theta - \frac{\pi}{2} + \varphi) - \cos^2(\theta - \frac{\pi}{2} + \varphi)] d\theta \right] = \frac{\pi}{ix}. \tag{2.71}$$

By using (2.54) and (2.71), the equation (2.51) reduces to

$$\begin{aligned}
 \frac{\partial I}{\partial x} &= \frac{-i}{4\pi^2 \mu} \left[ I_1 - \frac{\lambda^* + \mu}{\lambda^* + 2\mu} I_2 \right] = \frac{-i}{4\pi^2 \mu} \left[ \frac{2\pi}{ix} - \frac{\lambda^* + \mu}{\lambda^* + 2\mu} \left( \frac{\pi}{ix} \right) \right] \\
 &= \frac{-1}{2\pi \mu x} + \frac{\lambda^* + \mu}{\lambda^* + 2\mu} \left( \frac{1}{4\pi \mu x} \right).
 \end{aligned} \tag{2.72}$$

By integrating (2.72) with respect to  $x$ , one obtains

$$\begin{aligned}
 I &= \int \left( \frac{-1}{2\pi \mu x} + \frac{\lambda^* + \mu}{\lambda^* + 2\mu} \left( \frac{1}{4\pi \mu x} \right) \right) dx \\
 &= \frac{-1}{2\pi \mu} \int \frac{1}{x} dx + \frac{\lambda^* + \mu}{4(\lambda^* + 2\mu)\pi \mu} \int \frac{1}{x} dx
 \end{aligned}$$

$$\begin{aligned}
&= \frac{-1}{2\pi\mu} \ln x + \frac{\lambda^* + \mu}{4(\lambda^* + 2\mu)\pi\mu} \ln x = \left( \frac{-1}{2\pi\mu} + \frac{\lambda^* + \mu}{4(\lambda^* + 2\mu)\pi\mu} \right) \ln x \\
&= \left( \frac{-\lambda^* - 3\mu}{4(\lambda^* + 2\mu)\pi\mu} \right) \ln x. \quad (2.73)
\end{aligned}$$

Differentiating (2.49) with respect to  $\varphi$  yields

$$\begin{aligned}
\frac{\partial I}{\partial \varphi} &= \frac{1}{4\pi^2\mu} \left[ \lim_{R \rightarrow \infty} \left( \int_0^{2\pi} \int_0^R e^{-ikx \cos(\varphi-\theta)} \frac{-ikx(-\sin(\varphi-\theta))}{k} dk d\theta \right. \right. \\
&\quad \left. \left. - \frac{\lambda + \mu}{\lambda + 2\mu} \int_0^{2\pi} \int_0^R e^{-ikx \cos(\varphi-\theta)} \frac{-ikx(-\sin(\varphi-\theta))}{k} \cos^2 \theta dk d\theta \right) \right] \\
&= \frac{i}{4\pi^2\mu} \left[ \lim_{R \rightarrow \infty} \left( \int_0^{2\pi} \int_0^R e^{-ikx \cos(\varphi-\theta)} x \sin(\varphi-\theta) dk d\theta \right. \right. \\
&\quad \left. \left. - \frac{\lambda + \mu}{\lambda + 2\mu} \int_0^{2\pi} \int_0^R e^{-ikx \cos(\varphi-\theta)} x \sin(\varphi-\theta) \cos^2 \theta dk d\theta \right) \right]. \quad (2.74)
\end{aligned}$$

We calculate the integral

$$I_1 = \lim_{R \rightarrow \infty} \int_0^{2\pi} \int_0^R e^{-ikx \cos(\varphi-\theta)} x \sin(\varphi-\theta) dk d\theta, \quad (2.75)$$

by using the substitution

$$\theta \Rightarrow \theta_1 - \frac{\pi}{2} + \varphi, d\theta \Rightarrow d\theta_1,$$

$$\cos(\varphi-\theta) = \cos\left(\frac{\pi}{2} - \theta_1\right) = \sin \theta_1,$$

and

$$\sin(\varphi-\theta)=\sin\left(\frac{\pi}{2}-\theta_1\right)=\cos\theta_1,$$

where

$$\begin{aligned}\theta_1 &= \theta + \frac{\pi}{2} - \varphi, \\ \theta=0 &\Rightarrow \theta_1 = \frac{\pi}{2} - \varphi \\ \theta=2\pi &\Rightarrow \theta_1 = \frac{5\pi}{2} - \varphi.\end{aligned}\tag{2.76}$$

To simplify equation (2.75) to

$$\begin{aligned}I_1 &= \lim_{R \rightarrow \infty} \int_0^{2\pi} \int_0^R e^{-ikx \cos(\varphi-\theta)} x \sin(\varphi-\theta) dk d\theta = \lim_{R \rightarrow \infty} \int_{\frac{\pi}{2}-\varphi}^{\frac{5\pi}{2}-\varphi} \int_0^R e^{-ikx \sin\theta_1} x \cos\theta_1 dk d\theta_1 \\ &= \lim_{R \rightarrow \infty} \int_0^R \frac{e^{-ikx \sin\theta_1}}{-ik} \bigg|_{\frac{\pi}{2}-\varphi}^{\frac{5\pi}{2}-\varphi} dk = 0.\end{aligned}\tag{2.77}$$

In a similar way we can find

$$I_2 = \lim_{R \rightarrow \infty} \int_0^{2\pi} \int_0^R e^{-ikx \cos(\varphi-\theta)} x \sin(\varphi-\theta) \cos^2\theta dk d\theta.\tag{2.78}$$

Let's first find the integral

$$\int_0^{2\pi} e^{-ikx \cos(\varphi-\theta)} x \sin(\varphi-\theta) \cos^2 \theta \, d\theta, \quad (2.79)$$

which after using (2.76) and the substitution

$$\theta \Rightarrow \theta_1 - \frac{\pi}{2} + \varphi, d\theta \Rightarrow d\theta_1$$

$$\cos(\varphi-\theta) = \cos\left(\frac{\pi}{2} - \theta_1\right) = \sin \theta_1$$

$$\sin(\varphi-\theta) = \sin\left(\frac{\pi}{2} - \theta_1\right) = \cos \theta_1,$$

becomes

$$\int_{\frac{\pi}{2}-\varphi}^{\frac{5\pi}{2}-\varphi} e^{-ikx \sin \theta_1} \cos \theta_1 \cos^2\left(\theta_1 - \frac{\pi}{2} + \varphi\right) d\theta_1.$$

Since,

$$\begin{aligned} \cos^2\left(\theta_1 - \frac{\pi}{2} + \varphi\right) &= [\cos(\theta_1 - \frac{\pi}{2} + \varphi)]^2 = [\sin \theta_1 \cos \varphi + \cos \theta_1 \sin \varphi]^2 \\ &= \sin^2 \theta_1 \cos^2 \varphi + 2 \sin \theta_1 \cos \varphi \sin \varphi \cos \theta_1 + \sin^2 \varphi \cos^2 \theta_1, \end{aligned}$$

then

$$\begin{aligned}
& \int_{\frac{\pi}{2}-\varphi}^{\frac{5\pi}{2}-\varphi} e^{-ikx\sin\theta_1} \cos\theta_1 \cos^2\left(\theta_1 - \frac{\pi}{2} + \varphi\right) d\theta_1 \\
&= \int_{\frac{\pi}{2}-\varphi}^{\frac{5\pi}{2}-\varphi} e^{-ikx\sin\theta_1} \cos\theta_1 (\sin^2\theta_1 \cos^2\varphi + 2\sin\theta_1 \cos\varphi \sin\varphi \cos\theta_1 + \sin^2\varphi \cos^2\theta_1) d\theta_1 \\
&= \cos^2\varphi \int_{\frac{\pi}{2}-\varphi}^{\frac{5\pi}{2}-\varphi} e^{-ikx\sin\theta_1} \cos\theta_1 \sin^2\theta_1 d\theta_1 + 2\cos\varphi \sin\varphi \int_{\frac{\pi}{2}-\varphi}^{\frac{5\pi}{2}-\varphi} e^{-ikx\sin\theta_1} \sin\theta_1 \cos^2\theta_1 d\theta_1 \\
&\quad + \sin^2\varphi \int_{\frac{\pi}{2}-\varphi}^{\frac{5\pi}{2}-\varphi} e^{-ikx\sin\theta_1} \cos^3\theta_1 d\theta_1.
\end{aligned} \tag{2.80}$$

To calculate (2.80), we need to find the integral

$$\int_{\frac{\pi}{2}-\varphi}^{\frac{5\pi}{2}-\varphi} e^{-ikx\sin\theta_1} \cos\theta_1 \sin^2\theta_1 d\theta_1. \tag{2.81}$$

Using integration by parts with  $u = \sin^2\theta_1 \Rightarrow du = 2\sin\theta_1 \cos\theta_1 d\theta_1$  and

$$dv = \cos\theta_1 e^{-ikx\sin\theta_1} d\theta_1 \Rightarrow v = \frac{e^{-ikx\sin\theta_1}}{-ikx},$$

we find that



$$\begin{aligned}
\int_{\frac{\pi}{2}-\varphi}^{\frac{5\pi}{2}-\varphi} e^{-ikx\sin\theta_1} \cos\theta_1 \sin^2\theta_1 d\theta_1 &= \frac{e^{-ikx\sin\theta_1} \sin^2\theta_1}{-ikx} \Bigg|_{\frac{\pi}{2}-\varphi}^{\frac{5\pi}{2}-\varphi} + \frac{2}{ikx} \int_{\frac{\pi}{2}-\varphi}^{\frac{5\pi}{2}-\varphi} e^{-ikx\sin\theta_1} \cos\theta_1 \sin\theta_1 d\theta_1 \\
&= \frac{2}{ikx} \int_{\frac{\pi}{2}-\varphi}^{\frac{5\pi}{2}-\varphi} e^{-ikx\sin\theta_1} \cos\theta_1 \sin\theta_1 d\theta_1.
\end{aligned} \tag{2.82}$$

Again, we use integration by parts with  $u=\sin\theta_1 \Rightarrow du=\cos\theta_1 d\theta_1$ , and

$$dv=\cos\theta_1 e^{-ikx\sin\theta_1} d\theta_1 \Rightarrow v=\frac{e^{-ikx\sin\theta_1}}{-ikx}$$

to find

$$\begin{aligned}
\int_{\frac{\pi}{2}-\varphi}^{\frac{5\pi}{2}-\varphi} e^{-ikx\sin\theta_1} \cos\theta_1 \sin\theta_1 d\theta_1 &= \frac{e^{-ikx\sin\theta_1} \sin\theta_1}{-ikx} \Bigg|_{\frac{\pi}{2}-\varphi}^{\frac{5\pi}{2}-\varphi} - \int_{\frac{\pi}{2}-\varphi}^{\frac{5\pi}{2}-\varphi} \frac{e^{-ikx\sin\theta_1}}{-ikx} \cos\theta_1 d\theta_1 \\
&= \int_{\frac{\pi}{2}-\varphi}^{\frac{5\pi}{2}-\varphi} \frac{e^{-ikx\sin\theta_1}}{ikx} \cos\theta_1 d\theta_1 = \frac{1}{ikx} \frac{e^{-ikx\sin\theta_1}}{-ikx} \Bigg|_{\frac{\pi}{2}-\varphi}^{\frac{5\pi}{2}-\varphi} = 0.
\end{aligned} \tag{2.83}$$

It follows from (2.82) and (2.83) that

$$\int_{\frac{\pi}{2}-\varphi}^{\frac{5\pi}{2}-\varphi} e^{-ikx\sin\theta_1} \cos\theta_1 \sin^2\theta_1 d\theta_1 = 0. \tag{2.84}$$

In the same way, the integral

$$\int_{\frac{\pi}{2}-\varphi}^{\frac{5\pi}{2}-\varphi} e^{-ikx\sin\theta_1} \cos^3\theta_1 d\theta_1, \quad (2.85)$$

can be found by using integration by part with  $u=\cos^2\theta_1 \Rightarrow du=-2\sin\theta_1 \cos\theta_1 d\theta_1$  and

$$dv=\cos\theta_1 e^{-ikx\sin\theta_1} d\theta_1 \Rightarrow v=\frac{e^{-ikx\sin\theta_1}}{-ikx}, \text{ so that}$$

$$\begin{aligned} \int_{\frac{\pi}{2}-\varphi}^{\frac{5\pi}{2}-\varphi} e^{-ikx\sin\theta_1} \cos^3\theta_1 d\theta_1 &= \frac{e^{-ikx\sin\theta_1} \cos^2\theta_1}{-ikx} \Big|_{\frac{\pi}{2}-\varphi}^{\frac{5\pi}{2}-\varphi} - \frac{2}{ikx} \int_{\frac{\pi}{2}-\varphi}^{\frac{5\pi}{2}-\varphi} e^{-ikx\sin\theta_1} \cos\theta_1 \sin\theta_1 d\theta_1 \\ &= \frac{-2}{ikx} \int_{\frac{\pi}{2}-\varphi}^{\frac{5\pi}{2}-\varphi} e^{-ikx\sin\theta_1} \cos\theta_1 \sin\theta_1 d\theta_1 = 0. \end{aligned} \quad (2.86)$$

Finally, we need to calculate the integral

$$\int_{\frac{\pi}{2}-\varphi}^{\frac{5\pi}{2}-\varphi} e^{-ikx\sin\theta_1} \sin\theta_1 \cos^2\theta_1 d\theta_1, \quad (2.87)$$

by using integration by parts with  $u=\sin\theta_1 \cos\theta_1$  and  $dv=\cos\theta_1 e^{-ikx\sin\theta_1} d\theta_1$ , to give

$$du=(\cos^2\theta_1 - \sin^2\theta_1) d\theta_1, \quad v=\frac{e^{-ikx\sin\theta_1}}{-ikx}, \text{ and therefore,}$$

$$\begin{aligned}
\int_{\frac{\pi}{2}-\varphi}^{\frac{5\pi}{2}-\varphi} e^{-ikx\sin\theta_1} \sin\theta_1 \cos^2\theta_1 d\theta_1 &= \frac{\sin\theta_1 \cos\theta_1 e^{-ikx\sin\theta_1}}{-ikx} \Big|_{\frac{\pi}{2}-\varphi}^{\frac{5\pi}{2}-\varphi} - \int_{\frac{\pi}{2}-\varphi}^{\frac{5\pi}{2}-\varphi} \frac{e^{-ikx\sin\theta_1}}{-ikx} (\cos^2\theta_1 - \sin^2\theta_1) d\theta_1 \\
&= \int_{\frac{\pi}{2}-\varphi}^{\frac{5\pi}{2}-\varphi} \frac{e^{-ikx\sin\theta_1}}{ikx} (\cos^2\theta_1 - \sin^2\theta_1) d\theta_1.
\end{aligned} \tag{2.88}$$

Now,

$$\int_{\frac{\pi}{2}-\varphi}^{\frac{5\pi}{2}-\varphi} \frac{e^{-ikx\sin\theta_1}}{ikx} (\cos^2\theta_1 - \sin^2\theta_1) d\theta_1 = \frac{1}{ikx} \int_{\frac{\pi}{2}-\varphi}^{\frac{5\pi}{2}-\varphi} e^{-ikx\sin\theta_1} \cos^2\theta_1 d\theta_1 - \frac{1}{ikx} \int_{\frac{\pi}{2}-\varphi}^{\frac{5\pi}{2}-\varphi} e^{-ikx\sin\theta_1} \sin^2\theta_1 d\theta_1, \tag{2.89}$$

for which we can again use integration by parts with  $u = \cos\theta_1 \Rightarrow du = -\sin\theta_1 d\theta_1$  and

$$dv = \cos\theta_1 e^{-ikx\sin\theta_1} d\theta_1 \Rightarrow v = \frac{e^{-ikx\sin\theta_1}}{-ikx}, \text{ to obtain}$$

$$\begin{aligned}
\int_0^{2\pi} e^{-ikx\sin\theta_1} \cos^2\theta_1 d\theta_1 &= \frac{-e^{-ikx\sin\theta_1} \cos\theta_1}{ikx} \Big|_0^{2\pi} - \int_0^{2\pi} \left[ \frac{e^{-ikx\sin\theta_1}}{-ikx} (-\sin\theta_1 d\theta_1) \right] \\
&= - \int_0^{2\pi} \frac{e^{-ikx\sin\theta_1}}{ikx} \sin\theta_1 d\theta_1 \\
&= \frac{-1}{ikx} \int_0^{2\pi} e^{-ikx\sin\theta_1} \sin\theta_1 d\theta_1.
\end{aligned} \tag{2.90}$$

Let  $I^* = \int_0^{2\pi} e^{-ikx\sin\theta_1} d\theta_1 = 2\pi J_0(kx)$ . Then

$$\frac{\partial I^*}{\partial k} = \int_0^{2\pi} e^{-ikx\sin\theta_1} (-ix\sin\theta_1) d\theta_1$$

$$\begin{aligned}
\Rightarrow 2\pi x J'_0(kx) &= \int_0^{2\pi} e^{-ikx \sin \theta_1} (-ix \sin \theta_1) d\theta_1, \\
\Rightarrow \frac{-2\pi J'_0(kx)}{i} &= \int_0^{2\pi} e^{-ikx \sin \theta_1} \sin \theta_1 d\theta_1,
\end{aligned} \tag{2.91}$$

which can be substituted into equation (2.90) to give

$$\int_0^{2\pi} e^{-ikx \sin \theta_1} \cos^2 \theta_1 d\theta_1 = \frac{-1}{ikx} \left( \frac{-2\pi J'_0(kx)}{i} \right) = \frac{-2\pi J'_0(kx)}{kx}. \tag{2.92}$$

Evaluating the following integral yields

$$\begin{aligned}
\int_0^{2\pi} e^{-ikx \sin \theta_1} \sin^2 \theta_1 d\theta_1 &= \int_0^{2\pi} e^{-ikx \sin \theta_1} (1 - \cos^2 \theta_1) d\theta_1 \\
&= \int_0^{2\pi} e^{-ikx \sin \theta_1} d\theta_1 - \int_0^{2\pi} e^{-ikx \sin \theta_1} \cos^2 \theta_1 d\theta_1,
\end{aligned} \tag{2.93}$$

in which  $\sin^2 \theta = 1 - \cos^2 \theta$ . By substituting the definition of the Bessel function and equation (2.92) into equation (2.93), one obtains

$$\int_0^{2\pi} e^{-ikx \sin \theta_1} \sin^2 \theta_1 d\theta_1 = 2\pi J_0(kx) + \frac{2\pi J'_0(kx)}{kx}. \tag{2.94}$$

Substituting (2.92) and (2.94) into (2.89) gives

$$\begin{aligned}
\int_{\frac{\pi}{2}-\varphi}^{\frac{5\pi}{2}-\varphi} \frac{e^{-ikx\sin\theta_1}}{ikx} (\cos^2\theta_1 - \sin^2\theta_1) d\theta_1 &= \frac{1}{ikx} \left( \frac{-2\pi J'_0(kx)}{kx} \right) - \frac{1}{ikx} \left( 2\pi J_0(kx) + \frac{2\pi J'_0(kx)}{kx} \right) \\
&= \frac{-2\pi J'_0(kx)}{ik^2x^2} - \frac{1}{ikx} 2\pi J_0(kx) - \frac{2\pi J'_0(kx)}{ik^2x^2} \\
&= \frac{-4\pi J'_0(kx)}{ik^2x^2} - \frac{2\pi J_0(kx)}{ikx},
\end{aligned} \tag{2.95}$$

which after substituting (2.95) into (2.88) we get

$$\int_{\frac{\pi}{2}-\varphi}^{\frac{5\pi}{2}-\varphi} e^{-ikx\sin\theta_1} \sin\theta_1 \cos^2\theta_1 d\theta_1 = \frac{-4\pi J'_0(kx)}{ik^2x^2} - \frac{2\pi J_0(kx)}{ikx}. \tag{2.96}$$

Then the integral (2.80) becomes

$$\begin{aligned}
\int_0^{2\pi} e^{-ikx\cos(\varphi-\theta)} x \sin(\varphi-\theta) \cos^2\theta d\theta &= 2\cos\varphi \sin\varphi \int_{\frac{\pi}{2}-\varphi}^{\frac{5\pi}{2}-\varphi} e^{-ikx\sin\theta_1} \sin\theta_1 \cos^2\theta_1 d\theta_1 \\
&= 2\cos\varphi \sin\varphi \left[ \frac{-4\pi J'_0(kx)}{ik^2x^2} - \frac{2\pi J_0(kx)}{ikx} \right].
\end{aligned} \tag{2.97}$$

Rewriting (2.78) as

$$I_2 = \lim_{R \rightarrow \infty} \int_0^{2\pi} \int_0^R e^{-ikx\cos(\varphi-\theta)} x \sin(\varphi-\theta) \cos^2\theta dk d\theta$$

$$I_2 = \lim_{R \rightarrow \infty} \int_0^R 2x \cos \varphi \sin \varphi \left[ \frac{-4\pi J'_0(k)}{ik^2 x^2} - \frac{2\pi J_0(k)}{ikx} \right] dk, \quad (2.98)$$

the equation (2.98) can be solved by letting  $K=kx \Rightarrow dK=xdk$ . This gives

$$\begin{aligned} I_2 &= 2x \cos \varphi \sin \varphi \lim_{R \rightarrow \infty} \int_0^R \left[ \frac{-4\pi J'_0(K)}{iK^2} - \frac{2\pi J_0(K)}{iK} \right] \frac{dK}{x} \\ &= 4 \cos \varphi \sin \varphi \left( \frac{1}{2} \right) \\ &= 2 \cos \varphi \sin \varphi. \end{aligned} \quad (2.99)$$

Substituting (2.77) and (2.99) into (2.74), we have

$$\begin{aligned} \frac{\partial I}{\partial \varphi} &= \frac{-i}{4\pi^2 \mu} \left[ \frac{\lambda^* + \mu}{\lambda^* + 2\mu} \lim_{R \rightarrow \infty} \int_0^{2\pi} \int_0^R e^{-ikx \cos(\varphi - \theta)} x \sin(\varphi - \theta) \cos^2 \theta dk d\theta \right] \\ &= \frac{-i}{4\pi^2 \mu} \left( \frac{\lambda^* + \mu}{\lambda^* + 2\mu} \right) (2\pi i \cos \varphi \sin \varphi) \\ &= \frac{(\lambda^* + \mu)}{2\pi \mu (\lambda^* + 2\mu)} \cos \varphi \sin \varphi. \end{aligned} \quad (2.100)$$

Integrating (2.100) with respect to  $\varphi$  gives

$$\begin{aligned} I &= \int \frac{(\lambda^* + \mu)}{2\pi \mu (\lambda^* + 2\mu)} \cos \varphi \sin \varphi d\varphi \\ &= \frac{(\lambda^* + \mu)}{2\pi \mu (\lambda^* + 2\mu)} \left( \frac{\cos^2 \varphi}{2} \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{(\lambda^* + \mu)}{4\pi\mu(\lambda^* + 2\mu)} \cos^2 \varphi \\
&= \frac{(\lambda^* + \mu)}{4\pi\mu(\lambda^* + 2\mu)} \frac{x_1^2}{x^2},
\end{aligned} \tag{2.101}$$

in which

$$x_1 = x \cos \varphi \Rightarrow \frac{x_1^2}{x^2} = \cos^2 \varphi, \quad x = \sqrt{x_1^2 + x_2^2}.$$

Now, from (2.49) it follows that

$$\begin{aligned}
G_{11} &= \left( \frac{-\lambda^* - 3\mu}{4(\lambda^* + 2\mu)\pi\mu} \right) \ln x + \frac{(\lambda^* + \mu)}{4\pi\mu(\lambda^* + 2\mu)} \frac{x_1^2}{x^2} \\
&= \frac{\lambda^* + 3\mu}{4(\lambda^* + 2\mu)\pi\mu} \ln \frac{1}{x} + \frac{\lambda^* + \mu}{4\pi\mu(\lambda^* + 2\mu)} \frac{x_1^2}{x^2}.
\end{aligned} \tag{2.102}$$

Thus the Green's function for two-dimensional plane strain is

$$G_{\alpha\beta} = \left( \frac{-\lambda^* - 3\mu}{4(\lambda^* + 2\mu)\pi\mu} \right) \delta_{\alpha\beta} \ln x + \frac{(\lambda^* + \mu)}{4\pi\mu(\lambda^* + 2\mu)} \frac{x_\alpha x_\beta}{x^2}, \tag{2.103}$$

which can write (2.103) as

$$G_{\alpha\beta} = \frac{\lambda^* + 3\mu}{4(\lambda^* + 2\mu)\pi\mu} \delta_{\alpha\beta} \ln \frac{1}{x} + \frac{\lambda^* + \mu}{4\pi\mu(\lambda^* + 2\mu)} \frac{x_\alpha x_\beta}{x^2}, \tag{2.104}$$

or equivalently,

$$G_{\alpha\beta} = \frac{(3-4\nu)}{8(1-\nu)\pi\mu} \delta_{\alpha\beta} \ln \frac{1}{x} + \frac{1}{8\pi\mu(1-\nu)} \frac{x_\alpha x_\beta}{x^2}. \quad (2.105)$$

Notice that Green's function for two-dimensional plane stress can be obtained from those for plain strain by replacing  $E$  by  $\frac{E(1+2\nu)}{(1+\nu)^2}$  and  $\nu$  by  $\frac{\nu}{(1+\nu)}$ .

It follows from (2.1)

$$\begin{aligned} u_\alpha(x) &= G_{\alpha\gamma}(x-x') F_\gamma \\ &= \frac{\lambda^* + 3\mu}{4(\lambda^* + 2\mu)\pi\mu} \delta_{\alpha\gamma} F_\gamma \ln \frac{1}{x} + \frac{\lambda^* + \mu}{4\pi\mu(\lambda^* + 2\mu)} \frac{x_\alpha x_\gamma}{x^2} F_\gamma. \end{aligned} \quad (2.106)$$

Or equivalently,

$$u_\alpha(x) = \frac{(3-4\nu)}{8(1-\nu)\pi\mu} \delta_{\alpha\gamma} F_\gamma \ln \frac{1}{x} + \frac{1}{8\pi\mu(1-\nu)} \frac{x_\alpha x_\gamma}{x^2} F_\gamma. \quad (2.107)$$

Now, one can calculate the strain field by using the relation between the strain and the displacement as:

Differentiating (2.106) with respect to  $x_\beta$  gives



$$u_{\alpha,\beta} = \frac{-(\lambda^*+3\mu)x_\beta}{4(\lambda^*+2\mu)\pi\mu x^2} \delta_{\alpha\gamma} F_\gamma + \frac{\lambda^*+\mu}{4\pi\mu(\lambda^*+2\mu)} F_\gamma \left[ \frac{1}{x^2} (\delta_{\alpha\beta} x_\gamma + \delta_{\gamma\beta} x_\alpha) + x_\alpha x_\gamma (-2x^{-3} \frac{x_\beta}{x}) \right]. \quad (2.108)$$

Since

$$\begin{aligned} u_\beta(x) &= G_{\beta\gamma}(x-x') F_\gamma \\ &= \frac{\lambda^*+3\mu}{4(\lambda^*+2\mu)\pi\mu} \delta_{\beta\gamma} F_\gamma \ln \frac{1}{x} + \frac{\lambda^*+\mu}{4\pi\mu(\lambda^*+2\mu)} \frac{x_\beta x_\gamma}{x^2} F_\gamma, \end{aligned} \quad (2.109)$$

then differentiating (2.109) with respect to  $x_\alpha$  gives

$$u_{\beta,\alpha} = \frac{-(\lambda^*+3\mu)x_\alpha}{4(\lambda^*+2\mu)\pi\mu x^2} \delta_{\beta\gamma} F_\gamma + \frac{\lambda^*+\mu}{4\pi\mu(\lambda^*+2\mu)} F_\gamma \left[ \frac{1}{x^2} (\delta_{\alpha\beta} x_\gamma + \delta_{\gamma\alpha} x_\beta) + x_\beta x_\gamma (-2x^{-3} \frac{x_\alpha}{x}) \right]. \quad (2.110)$$

Adding (2.108) and (2.110) yields

$$\begin{aligned} u_{\alpha,\beta} + u_{\beta,\alpha} &= \frac{-(\lambda^*+3\mu)}{4(\lambda^*+2\mu)\pi\mu} \left[ \frac{x_\beta}{x^2} \delta_{\alpha\gamma} F_\gamma + \frac{x_\alpha}{x^2} \delta_{\beta\gamma} F_\gamma \right] + \frac{\lambda^*+\mu}{4\pi\mu(\lambda^*+2\mu)} F_\gamma \frac{1}{x^2} [\delta_{\alpha\beta} x_\gamma + \delta_{\gamma\beta} x_\alpha + \delta_{\alpha\beta} x_\gamma + \delta_{\gamma\alpha} x_\beta] \\ &\quad + \frac{(-4)(\lambda^*+\mu)}{4\pi\mu(\lambda^*+2\mu)} F_\gamma \frac{x_\alpha x_\beta x_\gamma}{x^4} \\ &= F_\gamma \frac{1}{x^2} x_\beta \delta_{\alpha\gamma} \left[ \frac{-(\lambda^*+3\mu)}{4\pi\mu(\lambda^*+2\mu)} + \frac{\lambda^*+\mu}{4\pi\mu(\lambda^*+2\mu)} \right] + \frac{x_\alpha}{x^2} \delta_{\beta\gamma} F_\gamma \left[ \frac{-(\lambda^*+3\mu)}{4\pi\mu(\lambda^*+2\mu)} + \frac{\lambda^*+\mu}{4\pi\mu(\lambda^*+2\mu)} \right] \\ &\quad + \frac{2(\lambda^*+\mu)}{4\pi\mu(\lambda^*+2\mu)} F_\gamma \frac{1}{x^2} \delta_{\alpha\beta} x_\gamma + \frac{(-4)(\lambda^*+\mu)}{4\pi\mu(\lambda^*+2\mu)} F_\gamma \frac{x_\alpha x_\beta x_\gamma}{x^4}. \end{aligned} \quad (2.111)$$

This leads to

$$\begin{aligned}
 \varepsilon_{\alpha\beta} &= \frac{1}{2} (u_{\alpha,\beta} + u_{\beta,\alpha}) \\
 &= \frac{-2\mu}{8\pi\mu(\lambda^*+2\mu)x^2} F_\gamma x_\beta \delta_{\alpha\gamma} + \frac{-2\mu}{8\pi\mu(\lambda^*+2\mu)x^2} \delta_{\beta\gamma} F_\gamma x_\alpha + \frac{2(\lambda^*+\mu)}{8\pi\mu(\lambda^*+2\mu)} F_\gamma \frac{1}{x^2} \delta_{\alpha\beta} x_\gamma \\
 &\quad + \frac{(-4)(\lambda^*+\mu)}{8\pi\mu(\lambda^*+2\mu)} F_\gamma \frac{x_\alpha x_\beta x_\gamma}{x^4} \\
 &= \frac{-2\mu}{8\pi\mu(\lambda^*+2\mu)x^2} F_\alpha x_\beta + \frac{-2\mu}{8\pi\mu(\lambda^*+2\mu)x^2} F_\beta x_\alpha + \frac{2(\lambda^*+\mu)}{8\pi\mu(\lambda^*+2\mu)x^2} F_\gamma \delta_{\alpha\beta} x_\gamma \\
 &\quad + \frac{(-4)(\lambda^*+\mu)}{8\pi\mu(\lambda^*+2\mu)} F_\gamma \frac{x_\alpha x_\beta x_\gamma}{x^4} \\
 &= \frac{-\mu}{4\pi\mu(\lambda^*+2\mu)x^2} [F_\alpha x_\beta + F_\beta x_\alpha - \frac{(\lambda^*+\mu)}{\mu} F_\gamma \delta_{\alpha\beta} x_\gamma + 2(\lambda^*+\mu) F_\gamma \frac{x_\alpha x_\beta x_\gamma}{2\mu x^2}]. \quad (2.112)
 \end{aligned}$$

The stress field can be found by Hooke's law as

$$\begin{aligned}
 \sigma_{\alpha\beta} &= C_{\alpha\beta\delta\gamma} \varepsilon_{\delta\gamma} \\
 &= \lambda^* \delta_{\alpha\beta} \varepsilon_{\gamma\gamma} + 2\mu \varepsilon_{\alpha\beta} \\
 &= \frac{-2\mu\lambda^* \delta_{\alpha\beta}}{4(\lambda^*+2\mu)\pi\mu x^2} x_\delta F_\delta + 2\mu \left( \frac{-\mu}{4\pi\mu(\lambda^*+2\mu)x^2} [F_\alpha x_\beta + F_\beta x_\alpha - \frac{(\lambda^*+\mu)}{\mu} F_\gamma \delta_{\alpha\beta} x_\gamma \right. \\
 &\quad \left. + 2(\lambda^*+\mu) F_\gamma \frac{x_\alpha x_\beta x_\gamma}{2\mu x^2} \right)]. \quad (2.113)
 \end{aligned}$$

After simplification, equation (2.113) becomes as follows

$$\begin{aligned} \sigma_{\alpha\beta} = & \frac{-\lambda^* \delta_{\alpha\beta}}{2(\lambda^* + 2\mu)\pi x^2} x_\delta F_\delta + \frac{-\mu}{2\pi(\lambda^* + 2\mu)x^2} [F_\alpha x_\beta + F_\beta x_\alpha - \frac{(\lambda^* + \mu)}{\mu} F_\gamma \delta_{\alpha\beta} x_\gamma \\ & + 2(\lambda^* + \mu) F_\gamma \frac{x_\alpha x_\beta x_\gamma}{\mu x^2}]. \end{aligned} \quad (2.114)$$

Or equivalently,

$$\sigma_{\alpha\beta} = \frac{-(1-2\nu)F_\alpha x_\beta}{4\pi(1-\nu)x^2} - \frac{(1-2\nu)F_\beta x_\alpha}{4\pi(1-\nu)x^2} + \frac{\delta_{\alpha\beta}(1-2\nu)F_\gamma x_\gamma}{4\pi(1-\nu)x^2} - \frac{F_\gamma x_\gamma x_\alpha x_\beta}{2\pi(1-\nu)x^4},$$

where

$$\varepsilon_{\gamma\gamma} = \frac{1}{2}(u_{\gamma,\gamma} + u_{\gamma,\gamma}) = u_{\gamma,\gamma},$$

and

$$\begin{aligned} u_{\gamma,\gamma} &= \frac{-(\lambda^* + 3\mu)x_\gamma}{4(\lambda^* + 2\mu)\pi\mu x^2} \delta_{\gamma\delta} F_\delta + \frac{\lambda^* + \mu}{4\pi\mu(\lambda^* + 2\mu)} F_\delta \left[ \frac{1}{x^2} (\delta_{\gamma\gamma} x_\delta + \delta_{\delta\gamma} x_\gamma) + x_\delta x_\gamma (-2x^{-3} \frac{x_\gamma}{x}) \right] \\ &= \frac{-(\lambda^* + 3\mu)x_\delta}{4(\lambda^* + 2\mu)\pi\mu x^2} F_\delta + \frac{\lambda^* + \mu}{4\pi\mu(\lambda^* + 2\mu)x^2} (2x_\delta F_\delta + x_\delta F_\delta - 2x_\delta F_\delta) \\ &= \frac{-(\lambda^* + 3\mu)x_\delta}{4(\lambda^* + 2\mu)\pi\mu x^2} F_\delta + \frac{\lambda^* + \mu}{4\pi\mu(\lambda^* + 2\mu)} \frac{x_\delta F_\delta}{x^2} \\ &= \frac{-2\mu}{4(\lambda^* + 2\mu)\pi\mu x^2} x_\delta F_\delta = \varepsilon_{\gamma\gamma}. \end{aligned} \quad (2.115)$$

Then  $u_{\gamma,\gamma} = \varepsilon_{\gamma\gamma}$ .

## 2.5 Principles of fracture mechanics and their implications for fatigue

### 2.5.1 Introduction

The section starts with a brief description of the theoretical fundamentals of the most commonly used mathematical models characterising the stress fields or the displacement field ahead of the crack tip. After this, the phenomenon of fatigue crack closure is described.

### 2.5.2 Linear elastic fracture mechanics

Linear elastic fracture mechanics uses the theory of elasticity to calculate the stress field near to the tip of a crack, assuming that the material is isotropic and linearly elastic. There are three basic modes of crack deformation which can be defined as the state of stress of material around part of a crack tip [2]. A classification corresponding to the three situations represented in figure 2.1 is offered by Irwin [5].

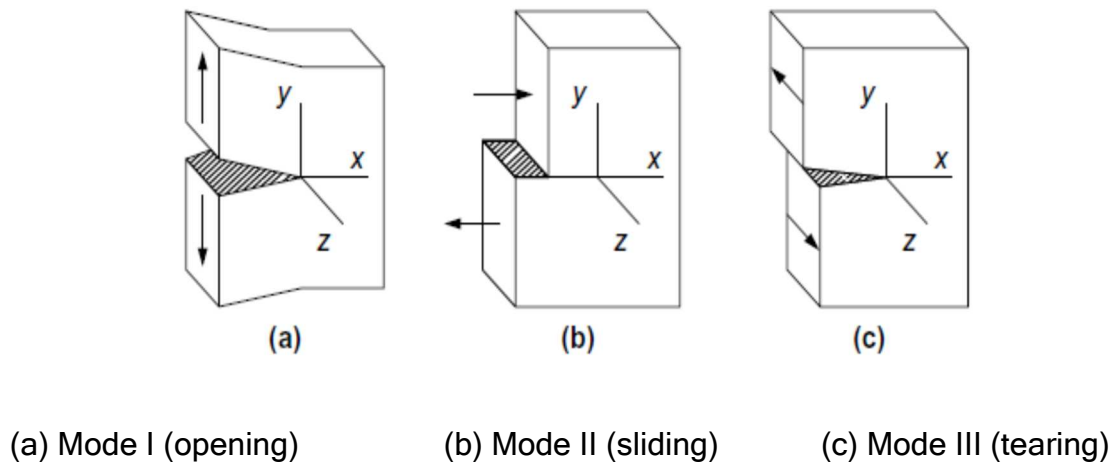


Figure 2.4 Crack opening modes [54]

In mode I, or opening mode, the body is loaded by a tensile stress normal to the plane of the crack, so that the crack surfaces are pulled apart in the  $y$  direction.

In mode II, or sliding mode, the body is loaded by a shear stress which acting parallel to the plane of the crack and perpendicular to the crack front.

In mode III, or tearing mode, the body is loaded by a shear stress which acting parallel to the plane of the crack surfaces and parallel to the crack front.

It should be noted that linear elastic fracture mechanics is only valid when the conditions for small scale yielding are satisfied and non-linear elastic plastic fracture mechanics are more suitable in case when large plastic deformation zones develop.

### 2.5.3 Crack tip stress analysis

Irwin [5] in the 1950's developed the stress intensity approach to analysing cracks.

The three basic modes of crack deformation can be expressed as [54]:

$$\sigma_{22} = \frac{K_I}{\sqrt{2\pi x_1}} + O(\sqrt{x_1}), \sigma_{12} = \sigma_{23} = 0$$

$$\sigma_{12} = \frac{K_{II}}{\sqrt{2\pi x_1}} + O(\sqrt{x_1}), \sigma_{22} = \sigma_{23} = 0$$

$$\sigma_{23} = \frac{K_{III}}{\sqrt{2\pi x_1}} + O(\sqrt{x_1}), \sigma_{22} = \sigma_{12} = 0.$$

where  $K_I$  is the Mode I stress intensity factor,  $K_{II}$  is the Mode II stress intensity factor and  $K_{III}$  is the Mode III stress intensity factor.

From these equations, it can be seen that the stresses have an inverse square root singularity at the crack tip, i.e. the region near to the crack tip is dominated by the singularity and the stress is proportional to  $1/\sqrt{x_1}$ .

The stresses near to the crack tip where there is no summation over  $x$  in the second term on the right hand side have the form [1]:

$$\sigma_{ij}(r,\theta) = \frac{K_I}{\sqrt{2\pi r}} \tilde{\sigma}_{ij}^I(\theta) + T \delta_{ix} \delta_{jx} + (\text{terms which vanish at crack tip}),$$

Where  $(r,\theta)$  are the cylindrical polar coordinates of a point with respect to the crack tip.  $K_I$  is called the stress intensity factor for mode I, which gives the magnitude of the elastic stress field and  $\tilde{\sigma}_{ij}^I(\theta)$  is a dimensionless quantity that depends on load and geometry. The quantity  $T$  corresponds to the so-called T-stress which is the second order term in the expansion.

The stress fields ahead of a crack tip for mode I and mode II are given by

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \begin{Bmatrix} 1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \\ 1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \\ \sin \frac{\theta}{2} \cos \frac{3\theta}{2} \end{Bmatrix} + \frac{K_{II}}{\sqrt{2\pi r}} \begin{Bmatrix} -\sin(\frac{\theta}{2})(2 + \cos(\frac{\theta}{2})\cos(\frac{3\theta}{2})) \\ \sin(\frac{\theta}{2})\cos(\frac{\theta}{2})\sin(\frac{3\theta}{2}) \\ \cos(\frac{\theta}{2})(1 - \sin(\frac{\theta}{2})\sin(\frac{3\theta}{2})) \end{Bmatrix}. \quad (2.116)$$

The corresponding crack tip displacement fields [1] are described by

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{K_I}{2\mu} \sqrt{\frac{r}{2\pi}} \begin{Bmatrix} \cos \frac{\theta}{2} (k-1+2\sin^2 \frac{\theta}{2}) \\ \sin \frac{\theta}{2} (k+1-2\cos^2 \frac{\theta}{2}) \end{Bmatrix} + \frac{K_{II}}{2\mu} \sqrt{\frac{r}{2\pi}} \begin{Bmatrix} \sin \frac{\theta}{2} (k+1+2\cos^2 \frac{\theta}{2}) \\ \cos \frac{\theta}{2} (k-1-2\sin^2 \frac{\theta}{2}) \end{Bmatrix}, \quad (2.117)$$

where  $\mu = \frac{E}{2(1+\nu)}$  is the shear modulus,  $E$  is the Young's modulus,  $\nu$  is Poisson's

ratio and  $k$  has been defined by Eq. (2.45), that is

$$k = \begin{cases} 3-4\nu & \text{for plane strain} \\ \frac{3-\nu}{1+\nu} & \text{for plane stress.} \end{cases}$$

From the equations (2.116), the stress near of a crack tip for mode I and mode II

when  $\theta=0$  has the following form:

$$\begin{aligned} \sigma_{11} = \sigma_{22} &= \frac{K_I}{\sqrt{2\pi r}}, \\ \sigma_{12} &= \frac{K_{II}}{\sqrt{2\pi r}} \end{aligned} \quad (2.118)$$

The stress intensity factor for mode I and mode II can therefore be defined as:

$$\begin{aligned} K_I &= \lim_{r \rightarrow 0} \{ \sqrt{2\pi r} \sigma_{22} \Big|_{\theta=0} \} \\ K_{II} &= \lim_{r \rightarrow 0} \{ \sqrt{2\pi r} \sigma_{12} \Big|_{\theta=0} \}. \end{aligned} \quad (2.119)$$



## 2.6 Fundamentals of fatigue crack closure phenomenon

The fatigue crack closure phenomenon is the subject of many experimental and analytical studies. This phenomenon was first studied and reported experimentally by Elber in 1970 [7]. He observed that a fraction of the crack tip is closed during the lower portion of the applied load cycle, i.e. the surfaces are apparently in contact (see, for instance, [5], [8] and [11]). Many research works have been published for the measurement and analysis of this phenomenon using different techniques and methods such as direct observation methods, indirect observation methods and methods based on compliance [5,37, 55, 60, 61 and 63]. Experimental observations published in the late 1970s established that Elber's mechanism was not the only cause of closure, and other types of closure phenomena may also influence the rate of fatigue crack advance. Other researchers considered the many forms of fatigue crack closure that can be caused by a change of mechanical, microstructural and environmental factors [1, 64 and 65]. These types of closure include plasticity induced crack closure, phase transformation induced crack closure, roughness induced crack closure, oxide induced crack closure and viscous fluid-induced crack closure, as shown in Figure 2.5. This range of mechanisms, along with controversy over whether plasticity-induced closure could exist in plane strain conditions or whether it was solely a plane stress phenomenon, led to a more appropriate term being used to describe the concept: crack tip shielding, i.e. a shielding of the crack tip from experiencing the full range of applied load and hence a reduction in crack growth rate [2, 58]. The mechanism of crack closure and shielding effects is complex and not fully understood and many of the results remain inaccurate and controversial [2]. This is due to the difficulty involved in quantifying the phenomenon and measuring its effect on the crack [6].

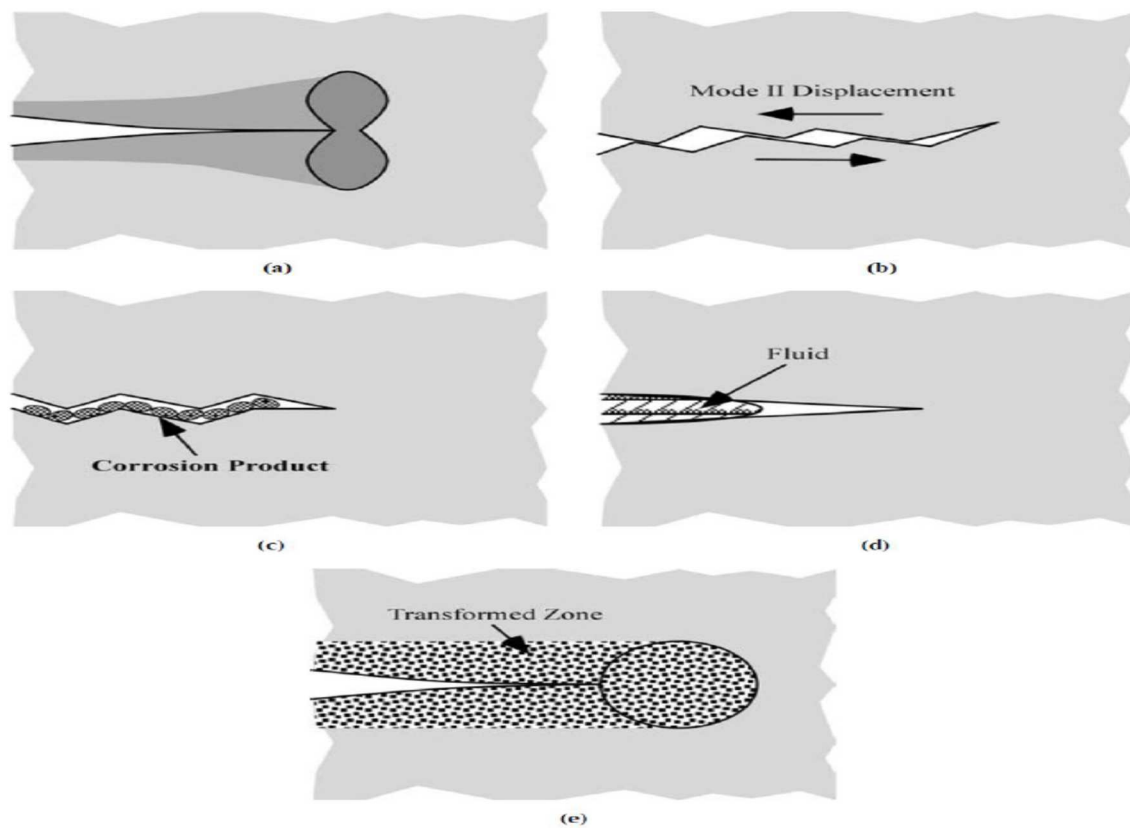


Figure 2.5 A schematic illustration of the different fatigue crack closure mechanisms. (a) Plasticity-induced crack closure, (b) oxide-induced crack closure, (c) roughness-induced crack closure, (d) viscous fluid-induced crack closure and (e) phase transformation-induced crack closure [8].

## 2.7 Description of the models for defining crack tip stress and displacement fields

There are several crack tip field models to describe stress or displacement and hence to predict plastic zone size and shape. The models consider in this work are the

Westergaard crack tip stress equations, and the recently developed CJP Mode for crack tip displacement.

## Westergaard equations

The stress field around the crack tip [9] based on the Westergaard equations is defined using the stress intensity factors (SIFs),  $K_I$  and  $K_{II}$ , the T - stress ( $-\sigma_{0x_1}$ ) more detailed information can be found in the Chapter 4. Crack tip stress fields are given by

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \begin{Bmatrix} 1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \\ 1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \\ \sin \frac{\theta}{2} \cos \frac{3\theta}{2} \end{Bmatrix} + \frac{K_{II}}{\sqrt{2\pi r}} \begin{Bmatrix} -\sin(\frac{\theta}{2})(2 + \cos(\frac{\theta}{2})\cos(\frac{3\theta}{2})) \\ \sin(\frac{\theta}{2})\cos(\frac{\theta}{2})\sin(\frac{3\theta}{2}) \\ \cos(\frac{\theta}{2})(1 - \sin(\frac{\theta}{2})\sin(\frac{3\theta}{2})) \end{Bmatrix} + \begin{Bmatrix} -\sigma_{0x_1} \\ 0 \\ 0 \end{Bmatrix}. \quad (2.120)$$

Crack tip displacement fields [1] are described by

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{K_I}{2\mu} \sqrt{\frac{r}{2\pi}} \begin{Bmatrix} \cos \frac{\theta}{2} (k-1+2\sin^2 \frac{\theta}{2}) \\ \sin \frac{\theta}{2} (k+1-2\cos^2 \frac{\theta}{2}) \end{Bmatrix} + \frac{K_{II}}{2\mu} \sqrt{\frac{r}{2\pi}} \begin{Bmatrix} \sin \frac{\theta}{2} (k+1+2\cos^2 \frac{\theta}{2}) \\ \cos \frac{\theta}{2} (k-1-2\sin^2 \frac{\theta}{2}) \end{Bmatrix} + \frac{\sigma_{0x_1}}{8\mu} r \begin{Bmatrix} (k+1)\cos\theta \\ (k-3)\sin\theta \end{Bmatrix}, \quad (2.121)$$

where  $\mu = \frac{E}{2(1+\nu)}$  is the shear modulus,  $E$  is the Young's modulus,  $\nu$  is Poisson's

ratio and  $k$  has been defined by Eq. (2.45), that is

$$k = \begin{cases} 3-4\nu & \text{for plane strain} \\ \frac{3-\nu}{1+\nu} & \text{for plane stress.} \end{cases}$$

## 2.8 Concluding comments

In this chapter, we have calculated the Green's function of the displacement field generated by point force acting at  $x'$  in an infinite body in two dimensions, which we will apply to Eshelby's method in Chapter 4 after we have re-expressed it in terms of Muskhelishvili's complex potential functions for 2D elasticity. In the last part of this chapter, we briefly addressed the phenomenon of fatigue crack closure.

## Chapter 3 Literature review

### 3.1 Introduction

Crack closure is a phenomenon associated with the mechanism of crack growth and which can be shown to affect fatigue crack growth rates. Elber [7] first introduced crack closure into fatigue crack growth analysis. After is, much research has been done concerning the crack closure effect using experimental studies, numerical analysis, and theoretical investigations [7, 16].

Eshelby [3] proposed the concept of a plastic ‘inclusion’ as a useful mathematical approach to dealing with part of a material that has undergone an ‘instantaneous’ change in properties, such as happens in a phase transformed zone or a region of plasticity.

This chapter has two main sections. It begins with a short historical overview of the fatigue crack closure phenomenon and a brief description of the different ways of measuring and estimating the crack closure effect. The second part of this chapter presents Eshelby’s idea and its use to find the solution for the stress, strain and displacement fields both inside and outside of a material inclusion assuming a permeant linear deformation of the material in the matrix.

### 3.2.1 Mechanisms and theoretical evaluation of crack closure

Elber [7, 16] was the first to discuss experimentally the phenomenon of fatigue crack closure in the early 1970s. After this, it used widely for the explanation of the influence of the load ratio,  $R$ , on fatigue crack growth behaviour in the near-threshold regime [22]. Recall that the load ratio (or stress ratio),  $R$ , is defined as the ratio of minimum to maximum load [22]. The degree of crack closure was higher at lower  $R$  [21] while it could be negligible at higher  $R$  [22].

The closure of the crack faces, even when the external load is tensile, has been called plasticity induced crack closure [6]. Plasticity induced crack closure is dependent on the external load, crack length, and material yielding properties [6].

Plasticity induced crack closure has also been used to explain the crack growth retardation subsequent to an overload (Suresh 1982), the mean stress effect ( $R$  ratio effect), short crack behaviour, and the existence of non-propagating cracks at notches [6].

Roughness induced crack closure mechanism has been considered by Walker and Beever [18]. This is caused by the contact of asperities on crack faces undergoing Mode II displacements.

The thin layer of fillings produced by the corrosion products and oxides residing on crack faces cause oxide induced crack closure found by Steward [19] and Ritchie et al. [20]. The increase of threshold stress intensity factor due to oxide induced crack closure was estimated by Suresh et al. [23].

Other areas in which the closure concept has been successfully applied in explaining observed crack growth behaviour are physically short fatigue cracks (where the limited crack wake reduces closure levels relative to long cracks, e.g. James and Smith in 1983 [66]).

Based on the postulated crack closure mechanisms, many models have been suggested to estimate crack closure effects. Budiansky and Hutchinson [24] established a theoretical approach by using Muskhelishvili's complex potentials on a Dugdale strip-yield model in 1978, which assumes that plastic yielding would occur in a narrow strip lying along the extension of the crack line under plane stress conditions [25].

For the analysis and assessment of fatigue crack closure phenomena, Mirzaei and Provan [26, 27] in 1992 proposed a rigid-insert crack closure model. This model was designed to account for the nonlinear elastic behaviour of a fatigue crack and estimated the combined effect of residual plastic stretches and corrosion debris on the closure behaviour of a fatigue crack by a hypothetical rigid insertion located in an ideal crack wake [2]. In the current work, we consider an elastic inclusion which gives more realistic with of the behaviour.

Plenty of numerical techniques to estimate crack closure have been developed. One of the numerical methods which deal with plasticity induced crack closure is an elastic plastic finite element analysis to estimate the crack opening and closing stresses first developed by Newman [28]. In the wake of Newman's work, many researchers have used and explained his technique to identify the sources of residual plastic deformation

producing the plastic wake and the existence of plasticity induced crack closure under plane strain conditions [29, 30]. Despite of the fact that finite element analysis for plasticity induced crack closure can be used for any crack geometry and loading condition, applying this technique to short cracks propagating under plane strain conditions is still difficult in practice because the element size has to be small enough to catch the small crack tip plastic zone [6].

Many researchers consider finite element analyses simulating plasticity induced fatigue crack closure in different two-dimensional configurations under plane strain or plane stress conditions [6], while there are a fewer efforts directed toward three-dimensional problems [29, 30].

Newman [31, 32] developed another numerical method which is a modification of the Dugdale strip-yield model for crack closure [25]. The model was based on the theory of Dugdale type cracks, to estimate the crack face contact stress by calculating the magnitude of the plastic wake left behind the crack tip.

### **3.2.2 Techniques for experimental measurement of fatigue crack closure**

The first reported experimental results that residual crack tip plastic deformations are left behind the crack tip was by Elber [16].



To determine when the crack is open and therefore experiencing a stress intensity at the tip, a wide range of experimental techniques have been applied to accurately capture displacements in the neighbourhood of the crack tip [15].

An indirect technique which can be employed to observe crack closure was developed by Elber [16]. This technique was based on Elber's measurements of the displacements at a number of positions in the neighbourhood of a crack. He noted that the variation of displacement with load, for loads below a certain level, was non-linear and inferred that the boundary conditions on the crack must be changing with load. This means that the crack must be partially closed for part of the load cycle [15].

A clip gauge, sometimes referred to as an 'Elber Gauge', is used to measure the variation of crack opening with load. The remote measurements of this sort show a very gradual change from non-linear to linear behaviour [15]. A typical curve is given in Figure 3.1 and shows the variation of displacement in the neighbourhood of a fatigue crack with load [15]. It can be seen that the crack is partially closed in the beginning, but it becomes fully open at a normalised load of approximately 0.4 and there is a corresponding gradual change in specimen stiffness [10, 15, 47].

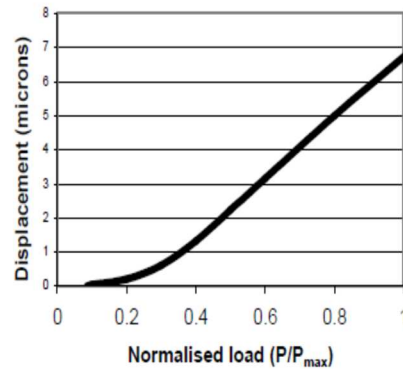


Figure 3.1 Typical 'compliance' curve of displacement in the neighbourhood of a fatigue crack with load [15]

The measuring of electrical resistance across a crack is another similar approach, which faces similar difficulties in identifying the precise point value of the crack tip opening load [47, 15, 17].

Other authors have suggested more direct measurements of crack closure such as the examination of the stress, strain, or displacement field in the neighbourhood of the crack tip in order to establish whether the crack is open [15].

The effect of specimen thickness on crack closure behaviour has been studied by Matos and Nowell [33]. They used three methods to assess crack closure, digital image correlation, back-face strain gauges, and crack-mouth clip gauges. In their study, a back-face strain gauge was suggested as the most suitable technique for determining effective crack closure for predicting crack growth rate [2].

Wei and James [34] used transmission photoelasticity methods to examine crack tip stress fields for polycarbonate specimens.

Pacey et al. [35] developed a mathematical model based on Muskhelishvili's complex potential functions [13] to evaluate the wake contact forces which were thought to affect the effective stress intensity factor range and also to describe the stress fields around the tip of a fatigue crack experiencing crack closure. They used photoelasticity combined with finite element methods and compliance measurements. Figure 3.2 shows the measuring of mode I stress intensity factor as closed circle, the mode II stress intensity factor as open circle and the theoretical mode I stress intensity factor neglecting closure as the dashed line [35].

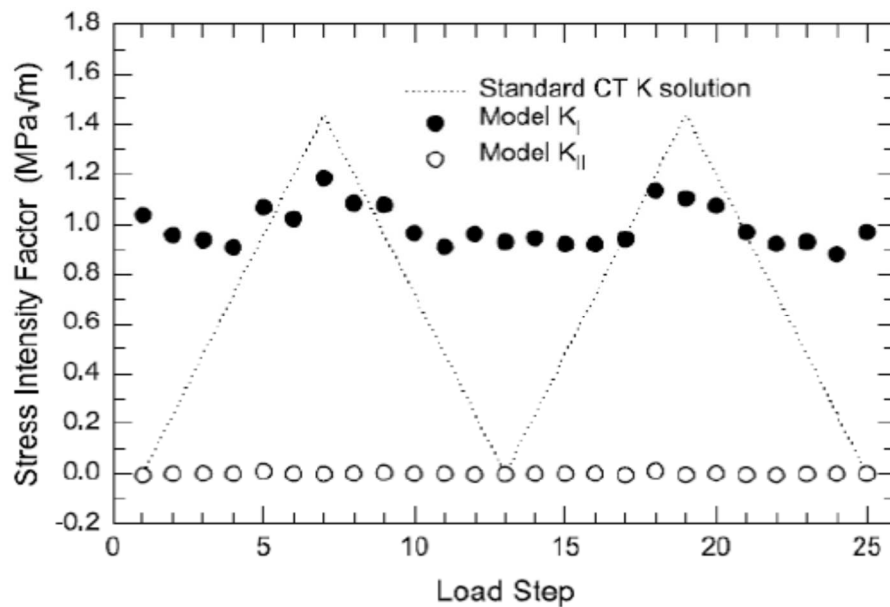


Figure 3.2 The variation measuring of the stress intensity factors during the two load cycles, (reproduced from [35])

James [36] in 1997 reviewed the potential sources of ambiguity arising from the use of different experimental techniques including differences between compliance methods and other systems. He also discussed the measurements obtained from the surface

and through-thickness measurements and positioning of the technique and disagreement on the importance and magnitude of plasticity-induced crack closure and in addition to sensitivity of the closure behaviour to materials, geometry, environment and test methodology [2, 12, 42].

As an important issue in developing techniques for engineering fatigue life prediction is that of finding how a growing crack influences plastic zone size and shape. Many analytical attempts have been proposed to estimate the role of the plastic zone such as Irwin (1960). He took some fraction of plastic zone size added to the true crack length to obtain an “effective” crack length. This leads to increasing the near tip stress field, which is inconsistent with the shielding effect of the plastic zone on the crack-tip field [38].

More recently several works (Chen, 2000, Fett and Munz, 2003, Ayatollahi and Zakeri, 2007, Christopher et al., 2007, Aliha et al., 2009), have shown broad agreement with Irwin (1957), and have confirmed that the T-stress is important for describing the state of stress and strain near to the crack tip [39]. The T-stress corresponds to the second order terms in the Williams and corresponds to a constant stress acting parallel to the crack [2].

The work [40] argues that the CJP model as a novel experimental methodology for the quantitative evaluation of the crack tip plastic zone size during fatigue crack growth. The CJP model seems to provide the best prediction of the crack tip plastic zone shape and size compared with either the Westergaard or Williams models.

The work [41] shows that the CJP model could be extended to deal with the case of mixed Mode I and Mode II loading, by include an addition of a new force parameter,  $F_T$  in Figure 3.3, to the model.

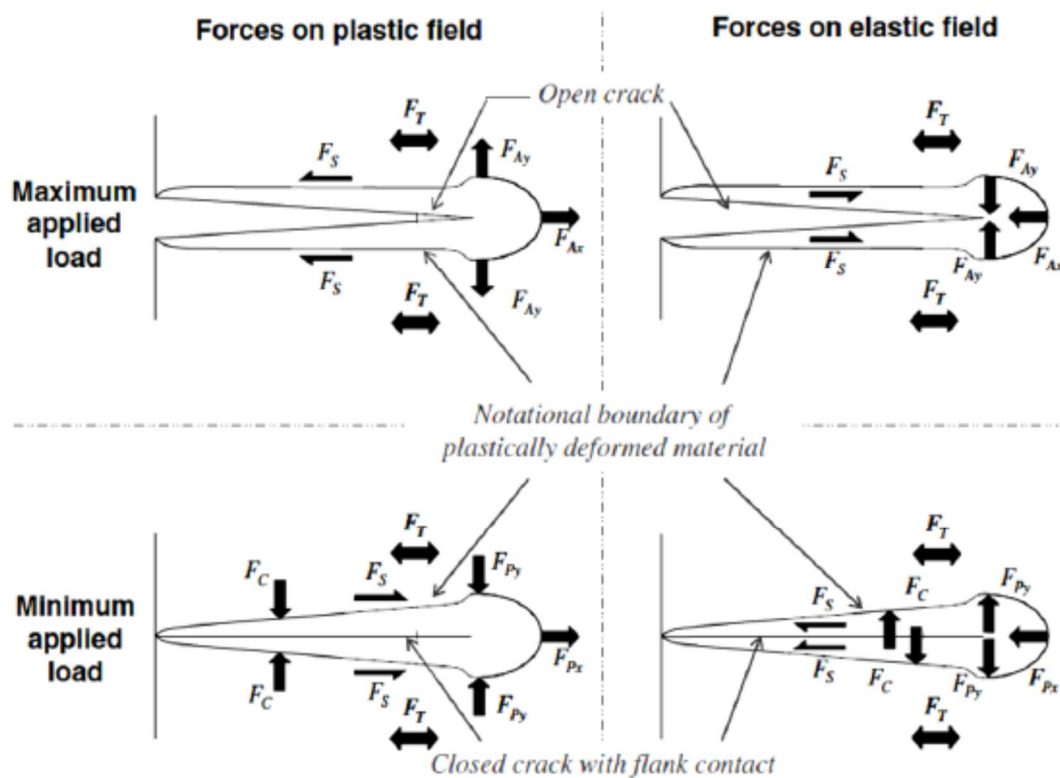


Figure 3.3. Schematic idealization of the forces acting at the interface of the plastic enclave and the surrounding elastic material in the CJP model [35].

Figure 3.4 shows the comparison of the experimental and CJP model predictions of plastic zone area for the high  $R = 0.6$  (CT1) and low  $R = 0.1$  (CT2) specimens and the plastic zone area obtained for the specimen tested at low  $R$ -ratio was smaller than the estimated value for the specimen tested at high  $R$ -ratio [40].

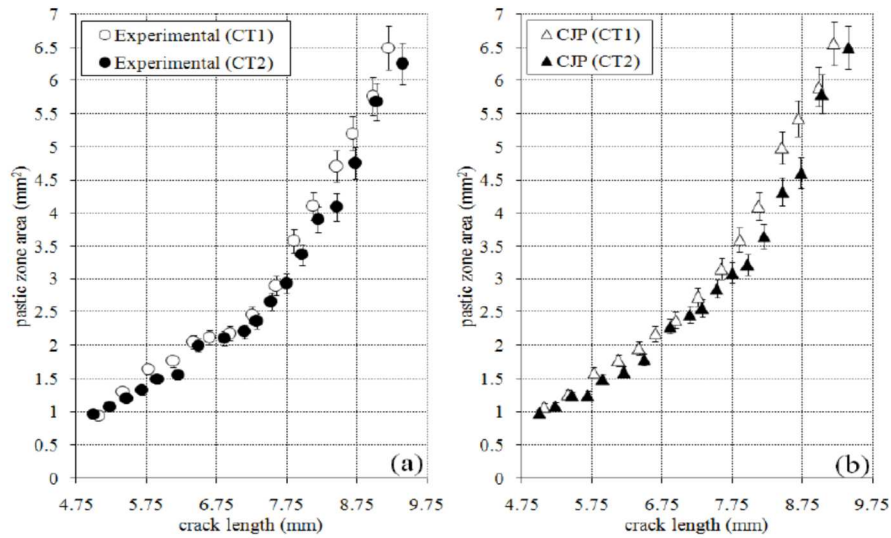


Figure 3.4 Comparison between the experimental and CJP predictions of plastic zone area, using the von Mises yield criterion [40].

### 3.3 Eshelby's inclusion and cracks

Inclusion problems tackle a variety of related questions including: non elastic constitutive equations; average elastic moduli and average thermal properties; transformation toughening; composites; dynamic effects sliding [43].

The so-called Eshelby's inclusion problem is to solve the stress, strain and displacement fields both in a subdomain which undergoes a permanent (inelastic) deformation (called the "inclusion") and its complement (called the "matrix") [50]. The strain under stress-free is called the eigenstrain. Eshelby was the first to introduce the notion of inclusion in (Eshelby, 1957) and then generalised by Mura in (Mura, 1982).

The solution of Eshelby (1957, 1961) has contributed significantly to the study of the effects of inhomogeneity in materials (e.g., Mura, 1987). Eshelby (1957) summarized his idea in the following steps: consider a region of an elastic body, the inclusion, which undergoes a change of size and shape that could be described by homogeneous strain in the absence of the constraint of the surrounding material and he found the resulting state of stress and strain in the inclusion and the surrounding matrix.

Eshelby solved this problem by series of conceptual steps involving cutting, transforming and re inserting the inclusion [44]. Eshelby's method was developed for a single ellipsoidal inclusion in an infinite elastic body (matrix). The resulting stress and strain in the inclusion are uniform when the inclusion undergoes a uniform stress-free transformation strain [see 38, 44 and 45].

Rudnicki (2011) described the approach of Eshelby for determining the stress and strain in regions in an infinite elastic body that undergoes a change of size or shape and the approach is extended to determine the stress and strain and displacement in regions of different elastic properties. He also discussed the relation of Eshelby's approach to singular solutions in elasticity and different integral forms for the solutions [44, 46]. To cite only a few examples the interaction of two ellipsoidal inclusions (Moschovidis and Mura 1975), the behaviour of hybrid composites (Taya and Chou 1981) and short fiber reinforced composites (Withers et al. 1989), the calculation of the stress fields inside a non-ellipsoidal inclusion which are not uniform (Johanson et al. 1980) [51].

In a part of the material which has undergone an 'instantaneous' change in properties, such as in a phase transformed zone or a region of plasticity, Eshelby's approach (Slaughter [48]) was proposed as a convenient mathematical approach to understanding the influences on the applied elastic stress field of the plastic enclave that is generated around a growing fatigue crack .

The paper [49] dealt with the relation between the inclusion ahead of a mode I crack tip and the crack tip stress intensity factor for various inclusion shapes and moduli, which assumed that the crack-inclusion separation, and the size of the inclusion are small compared with the length of the crack. In this study, we will talk about the effects elastic and plastic the crack with the inclusion next to the tip of crack in 2 dimensional plane stress case.

The Eshelby equivalent inclusion approach (Withers et al., 1989; Eshelby, 1957) is used to give a theoretical basis for numerical analysis (see [49, 56, 57, 62]).

Based on Eshelby equivalent inclusion approach, Li and Duan (2002) established similarity between a plastically deformed inclusion ahead of a crack tip and a transformed inclusion. They demonstrated that the plastic zone around crack tip can be identified with a transformed inclusion by means of Eshelby equivalent inclusion which was evaluated by using the theory of transformation toughening [38]. Figure 3.4 illustrates the simulants between the transformation toughening (a) and the plastic zone toughening (b) (see [38]). It considers on the crack tip and it does not seem to fully tack account of the behaviour along the crack flanks.



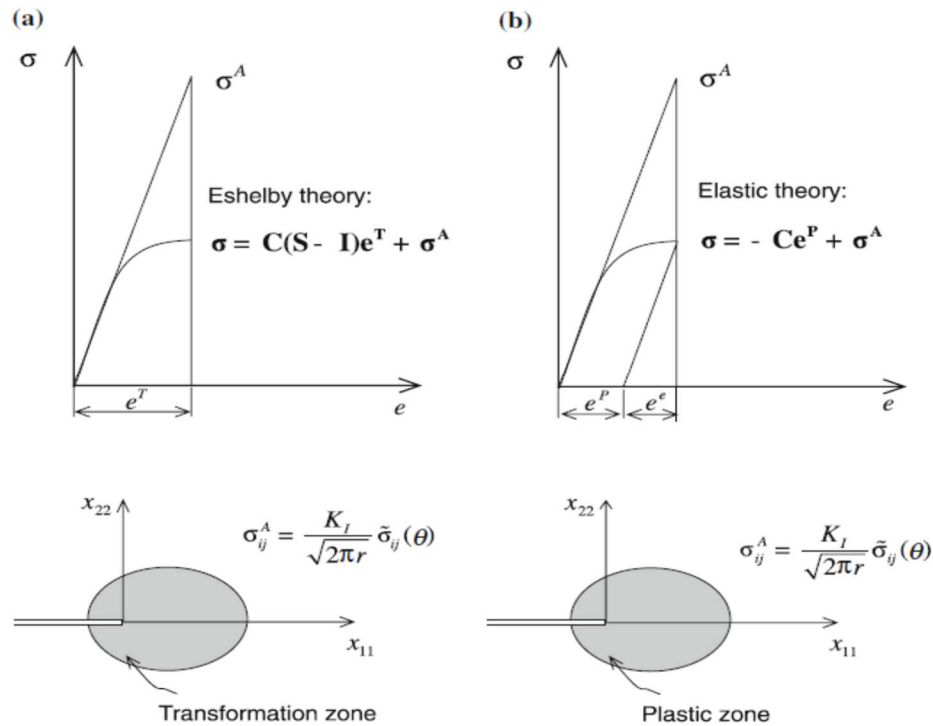


Figure 3.5 illustrating the similitude between the transformations toughening (a) and the plastic zone toughening (b) (reproduced from [38]).

### 3.4 Concluding comments

This chapter has described the importance of the crack closure concept for the prediction of fatigue crack growth. Numerical simulations have been used to complement as analytical and experimental approaches for the study of fatigue crack closure such as finite element methods and boundary element methods. There are many issues which make finite element methods very difficult to apply, such as mesh refinement, crack face contact, required computational effort, etc. [35].

Plasticity induced crack closure is the only crack closure mechanism which can be analytically modelled without involving the uncertainty of microstructural effects. Finite element techniques and Newman's modified Dugdale strip yield model are two approaches to estimate plasticity induced crack closure [6].

There are several experimental methods for measuring crack closure that have been proposed to evaluate crack closure effects in the past forty years such as direct observation of the crack closure at the crack tip, which include for example scanning electron microscopy, replica techniques, photography, and optical microscopy [52]. There are indirect methods, which are based on fatigue crack growth. The Compliance-based methods include strain gauges, clip gauges, and laser extensometer. Then the more commonly used methods because of their simplicity and relatively low cost [2, 52].

A new four-parameter photoelastic model has been proposed by Christopher et al [14]. This model was designed to more appropriately describe the shielding effect of the plastic zones ahead of the crack tip and along the crack wake. The model was employed in some studies to calculate from experimental data both the stress intensity factors and T-stress. [2].

In spite of the widely used and the important role which the plastic zone plays in the fracture process, still there is considerable disagreement and a well-recognized mechanical model for quantitative assessment of the amount of closure of the fracture has not yet been established [53].

Several experimental and theoretical works on cracks deal with the 2D (plane strain or plane stress) problem which can be solved using Eshelby's approach.

In this thesis, the approach of Eshelby is used as a basis for representing the deformation due to crack closure. The approach is described in more detail in the next chapters.

## Chapter 4 Complex potential representation of problem

### 4.1 Introduction

The stress field around the crack tip is one of the main factors determining the growth of cracks in a solid. There are many methods that are employed to obtain stresses and displacements in cracked bodies. Some of these methods are analytic ones, such as the complex potential function method and the integral transform method and others are numerical ones, such as the finite element method [54]. This chapter will present the complex potential function method and will use it to analyse the stresses and displacements around crack tips.

### 4.2 Method of complex potential for plane elasticity (The Kolosov-Muskhelishvili formulas)

The complex potential function method by Kolosov-Muskhelishvili [13] is one of the most useful mathematical methods for plane elasticity, and is used frequently for finding the solution to two-dimensional crack problems. We will give in this section a brief overview of the general formulation of the Kolosov-Muskhelishvili complex potentials.

#### 4.2.1 Complex potential representation of the Airy stress function

According to plane stress/ strain problems, the Airy stress function  $\Phi(X_1, X_2)$  defined as [48]

$$\sigma_{11}=\Phi_{,22}, \sigma_{22}=\Phi_{,11} \text{ and } \sigma_{12}=-\Phi_{,12}. \quad (4.1)$$

The Airy stress function  $\Phi$  is biharmonic ( $\nabla^4\Phi=0$ ) in a region  $S$  when the body forces are zero. Such a  $\Phi$  can be expressed as

$$\Phi=\text{Re}\{\bar{z}\varphi(z)+\Psi(z)\}, \quad (4.2)$$

where  $\varphi(z)$  and  $\Psi(z)$  are analytic functions in the region  $S$  and  $z=X_1+iX_2$  is a complex variable with conjugate  $\bar{z}=X_1-iX_2$ , see [54].

Substituting (4.2) in (4.1), we get,

$$\begin{aligned} \sigma_{11} &= \text{Re}\{-\bar{z}\varphi''(z)+2\varphi'(z)-\Psi''(z)\}, \\ \sigma_{22} &= \text{Re}\{\bar{z}\varphi''(z)+2\varphi'(z)+\Psi''(z)\}, \\ \sigma_{12} &= \text{Im}\{\bar{z}\varphi''(z)+\Psi''(z)\}. \end{aligned} \quad (4.3)$$

One can combine the equations (4.3) to obtain the relations

$$T=\sigma_{11}+\sigma_{22}=4\text{Re}\{\varphi'\}, \quad (4.4)$$

and

$$\Sigma=\sigma_{22}-\sigma_{11}+2i\sigma_{12}=2(\bar{z}\varphi''+\psi'), \quad (4.5)$$

where  $\psi(z)=\Psi'(z)$ . These analytic functions give representations of the stresses which satisfy the plane stress and plane strain equations.

We now address the complex potential representation of displacements. The relation between stresses and strains in equation (2.13) can be rewritten as

$$\sigma_{\alpha\beta} = 2\mu \left[ \varepsilon_{\alpha\beta} + \frac{3-k}{2(k-1)} \varepsilon_{\gamma\gamma} \delta_{\alpha\beta} \right], \quad (4.6)$$

where,

$$k = \begin{cases} 3-4\nu & \text{for plane strain} \\ \frac{3-\nu}{1+\nu} & \text{for plane stress} \end{cases}. \quad (4.7)$$

Using the formula above, and equation (2.5), we get

$$\sigma_{11} + \sigma_{22} = \frac{4\mu}{k-1} (u_{1,1} + u_{2,2}). \quad (4.8)$$

Therefore, from (4.4), we have

$$\mu(u_{1,1} + u_{2,2}) = (k-1) \operatorname{Re}\{\varphi'\}. \quad (4.9)$$

By using (4.6), we obtain

$$\sigma_{22} - \sigma_{11} + 2i\sigma_{12} = 2\mu[u_{2,2} - u_{1,1} + i(u_{1,2} + u_{2,1})], \quad (4.10)$$

and from (4.5), it follows that

$$\mu[u_{2,2} - u_{1,1} + i(u_{1,2} + u_{2,1})] = \bar{z}\varphi'' + \psi'. \quad (4.11)$$

From (4.9) and (4.11) we obtain

$$2\mu u_{1,1} = \operatorname{Re}\{(k-1)\varphi' - \bar{z}\varphi'' - \psi'\}, \quad (4.12)$$

and

$$2\mu u_{2,2} = \operatorname{Re}\{(k-1)\varphi' + \bar{z}\varphi'' + \psi'\}. \quad (4.13)$$

Equating the imaginary parts of both sides of (4.11), we have

$$\mu(u_{1,2} + u_{2,1}) = \operatorname{Im}\{\bar{z}\varphi'' + \psi'\}. \quad (4.14)$$

By integrating (4.12) and (4.13) with respect to  $X_1$  and  $X_2$  respectively, we obtain

$$2\mu u_1 = \operatorname{Re}\{(k-1)\varphi - \bar{z}\varphi' + \varphi - \psi\} + \zeta_1(X_2) = \operatorname{Re}\{k\varphi - \bar{z}\varphi' - \psi\} + \zeta_1(X_2), \quad (4.15)$$

$$2\mu u_2 = \operatorname{Re}\{-i(k-1)\varphi - i\bar{z}\varphi' - i\varphi - i\psi\} + \zeta_2(X_1) = \operatorname{Im}\{k\varphi + \bar{z}\varphi' + \psi\} + \zeta_2(X_1), \quad (4.16)$$

Where  $\zeta_1$  and  $\zeta_2$  are an arbitrary real-valued function of  $X_2$  and  $X_1$  respectively.

Differentiating equation (4.15) with respect to  $X_2$  we get,

$$\begin{aligned}
2\mu u_{1,2} &= \operatorname{Re} \left\{ k\phi' \frac{dz}{dX_2} - \bar{z}\phi'' \frac{dz}{dX_2} - \phi' \frac{d\bar{z}}{dX_2} - \psi' \frac{dz}{dX_2} \right\} + \zeta_1'(X_2) \\
&= \operatorname{Re} \{ k\phi'(i) - \phi'(-i) - \bar{z}\phi''(i) - \psi'(i) \} + \zeta_1'(X_2) \\
&= \operatorname{Re} \{ (k+1)\phi'(i) - \bar{z}\phi''(i) - \psi'(i) \} + \zeta_1'(X_2) \\
&= \operatorname{Im} \{ -(k+1)\phi' + \bar{z}\phi'' + \psi'(i) \} + \zeta_1'(X_2),
\end{aligned} \tag{4.17}$$

and differentiating equation (4.16) with respect to  $X_1$  we get,

$$\begin{aligned}
2\mu u_{2,1} &= \operatorname{Im} \left\{ k\phi' \frac{dz}{dX_1} + \bar{z}\phi'' \frac{dz}{dX_1} + \phi' \frac{d\bar{z}}{dX_1} + \psi' \frac{dz}{dX_1} \right\} + \zeta_2'(X_1) \\
&= \operatorname{Im} \{ k\phi' + \bar{z}\phi'' + \phi' + \psi' \} + \zeta_2'(X_1) \\
&= \operatorname{Im} \{ (k+1)\phi' + \bar{z}\phi'' + \psi' \} + \zeta_2'(X_1).
\end{aligned} \tag{4.18}$$

It follows from (4.17) and (4.18) that

$$2\mu(u_{1,2} + u_{2,1}) = 2\operatorname{Im} \{ \bar{z}\phi'' + \psi' \} + \zeta_1'(X_2) + \zeta_2'(X_1). \tag{4.19}$$

By comparing (4.19) with (4.14) one can see that

$$\zeta_1'(X_2) = -\zeta_2'(X_1) = \gamma \Rightarrow \zeta_1(X_2) = \alpha + \gamma X_2, \quad \zeta_2(X_1) = \beta - \gamma X_1,$$

where  $\gamma, \alpha$  and  $\beta$  are real constants. Since  $\operatorname{Re}\{\phi\} = \operatorname{Re}\{\bar{\phi}\}$ ,  $\operatorname{Im}\{\phi\} = -\operatorname{Im}\{\bar{\phi}\}$  and

$\bar{\phi} = \overline{\phi(z)} = \overline{\phi(\bar{z})}$ , then equations (4.15) and (4.16) can be written as



$$2\mu u_1 = \operatorname{Re}\{k\bar{\varphi} - \bar{z}\varphi' - \psi\}, \quad (4.20)$$

$$2\mu u_2 = \operatorname{Im}\{-k\bar{\varphi} + \bar{z}\varphi' + \psi\}.$$

Writing (4.20) in complex form, we have

$$2\mu(u_1 - i u_2) = -\bar{z}\varphi' + k\bar{\varphi} - \psi, \quad (4.21)$$

where  $k$  is defined by (4.7). Equations (4.4), (4.5), and (4.21) are the Kolosov-Muskhelishvili formulas.

#### 4.2.2 The complex potential representation of the components of stress and displacements induced by point forces

##### 4.2.2.1 The complex potential representation of the components of stress induced by point forces

We want to know the most general expressions for the complex stress functions  $\varphi$  and  $\psi$  in terms of the components of the stress  $\sigma_{ij}$  from a point force found in section 2.4.3.

Now, from (2.114) we have

$$\sigma_{\alpha\beta} = \frac{-\mu F_\alpha x_\beta}{2\pi(\lambda^* + 2\mu)x^2} - \frac{\mu F_\beta x_\alpha}{2\pi(\lambda^* + 2\mu)x^2} + \frac{\mu \delta_{\alpha\beta} F_\gamma x_\gamma}{2\pi(\lambda^* + 2\mu)x^2} - \frac{2\mu(\lambda^* + \mu)F_\gamma x_\gamma x_\alpha x_\beta}{2\pi\mu(\lambda^* + 2\mu)x^4},$$

or equivalently,

$$\sigma_{\alpha\beta} = \frac{-(1-2\nu)F_{\alpha}x_{\beta}}{4\pi(1-\nu)x^2} - \frac{(1-2\nu)F_{\beta}x_{\alpha}}{4\pi(1-\nu)x^2} + \frac{\delta_{\alpha\beta}(1-2\nu)F_{\gamma}x_{\gamma}}{4\pi(1-\nu)x^2} - \frac{F_{\gamma}x_{\gamma}x_{\alpha}x_{\beta}}{2\pi(1-\nu)x^4}.$$

Therefore,

$$\sigma_{11} = \frac{-\mu F_1 x_1 + \mu F_2 x_2}{2\pi(\lambda^* + 2\mu)x^2} - \frac{2(\mu + \lambda^*)F_1 x_1^3}{2\pi(\lambda^* + 2\mu)x^4} - \frac{2(\mu + \lambda^*)F_2 x_2 x_1^2}{2\pi(\lambda^* + 2\mu)x^4}, \quad (4.22)$$

and

$$\sigma_{22} = \frac{-\mu F_2 x_2 + \mu F_1 x_1}{2\pi(\lambda^* + 2\mu)x^2} - \frac{2(\mu + \lambda^*)F_2 x_2^3}{2\pi(\lambda^* + 2\mu)x^4} - \frac{2(\mu + \lambda^*)F_1 x_1 x_2^2}{2\pi(\lambda^* + 2\mu)x^4}. \quad (4.23)$$

We can combine (4.22) and (4.23) to give

$$T = \sigma_{11} + \sigma_{22} = \frac{-2(\mu + \lambda^*)F_1 x_1^3 - 2(\mu + \lambda^*)F_2 x_2^3}{2\pi(\lambda^* + 2\mu)x^4} + \frac{-2(\mu + \lambda^*)F_2 x_2 x_1^2 - 2(\mu + \lambda^*)F_1 x_1 x_2^2}{2\pi(\lambda^* + 2\mu)x^4}. \quad (4.24)$$

Simplifying and evaluating (4.24), we get

$$\begin{aligned}
T = \sigma_{11} + \sigma_{22} &= \frac{-2(\mu + \lambda^*)[F_1 x_1^3 + F_2 x_2^3]}{2\pi(\lambda^* + 2\mu)x^4} + \frac{-2(\mu + \lambda^*)[F_2 x_2 x_1^2 + F_1 x_1 x_2^2]}{2\pi(\lambda^* + 2\mu)x^4} \\
&= \frac{-2(\mu + \lambda^*)[F_1 x_1^3 + F_2 x_2^3 + F_2 x_2 x_1^2 + F_1 x_1 x_2^2]}{2\pi(\lambda^* + 2\mu)x^4} \\
&= \frac{-(\mu + \lambda^*)[x_1^2(F_1 x_1 + F_2 x_2) + x_2^2(F_2 x_2 + F_1 x_1)]}{\pi(\lambda^* + 2\mu)x^4} \\
&= \frac{-(\mu + \lambda^*)[(F_1 x_1 + F_2 x_2)(x_1^2 + x_2^2)]}{\pi(\lambda^* + 2\mu)x^4}.
\end{aligned} \tag{4.25}$$

We rewrite the equation (4.25) as

$$\begin{aligned}
T &= \frac{-(\mu + \lambda^*)}{\pi(\lambda^* + 2\mu)} \frac{\operatorname{Re}\{F\bar{z}\}|z|^2}{|z|^4} = \frac{-(\mu + \lambda^*)}{\pi(\lambda^* + 2\mu)} \operatorname{Re}\left\{\frac{F\bar{z}}{|z|^2}\right\} \\
&= \frac{-(\mu + \lambda^*)}{\pi(\lambda^* + 2\mu)} \operatorname{Re}\left\{\frac{F}{z}\right\},
\end{aligned} \tag{4.26}$$

where  $z = x_1 + ix_2$ ,  $\bar{z} = x_1 - ix_2$ , and  $F = F_1 + iF_2$ . By using equation (4.4),  $T = \sigma_{11} + \sigma_{22} = 4\operatorname{Re}\{\varphi'\}$ ,

and from equation (4.26) we have

$$4\operatorname{Re}\{\varphi'\} = \frac{-(\mu + \lambda^*)}{\pi(\lambda^* + 2\mu)} \operatorname{Re}\left\{\frac{F}{z}\right\}, \tag{4.27}$$

which we can write as

$$\operatorname{Re}\{\varphi'\} = \frac{-(\mu + \lambda^*)}{4\pi(\lambda^* + 2\mu)} \operatorname{Re}\left\{\frac{F}{z}\right\}. \tag{4.28}$$

Integrating (4.28) gives

$$\varphi = \frac{-(\mu + \lambda^*)}{4\pi(\lambda^* + 2\mu)} F \ln(z), \quad (4.29)$$

or, equivalently in terms of  $v$ ,

$$\varphi = \frac{-F \ln(z)}{8\pi(1-v)}. \quad (4.30)$$

Subtracting (4.23) from (4.22), one obtains

$$\begin{aligned} \sigma_{22} - \sigma_{11} = & \frac{-\mu F_2 x_2 + \mu F_1 x_1}{2\pi(\lambda + 2\mu)x^2} - \frac{2(\mu + \lambda)F_2 x_2^3}{2\pi(\lambda + 2\mu)x^4} - \frac{2(\mu + \lambda)F_1 x_1 x_2^2}{2\pi(\lambda + 2\mu)x^4} + \frac{\mu F_1 x_1 - \mu F_2 x_2}{2\pi(\lambda + 2\mu)x^2} \\ & + \frac{2(\mu + \lambda)F_1 x_1^3}{2\pi(\lambda + 2\mu)x^4} + \frac{2(\mu + \lambda)F_2 x_2 x_1^2}{2\pi(\lambda + 2\mu)x^4}, \end{aligned} \quad (4.31)$$

or equivalently,

$$\begin{aligned} \sigma_{22} - \sigma_{11} = & \frac{-2\mu F_2 x_2 + 2\mu F_1 x_1}{2\pi(\lambda^* + 2\mu)x^2} + \frac{2(\mu + \lambda^*)[F_1 x_1^3 - F_2 x_2^3]}{2\pi(\lambda^* + 2\mu)x^4} + \frac{2(\mu + \lambda^*)[F_2 x_2 x_1^2 - F_1 x_1 x_2^2]}{2\pi(\lambda^* + 2\mu)x^4} \\ = & \frac{-2\mu F_2 x_2 + 2\mu F_1 x_1}{2\pi(\lambda^* + 2\mu)x^2} + \frac{2(\mu + \lambda^*)[F_1 x_1^3 - F_2 x_2^3 + F_2 x_2 x_1^2 - F_1 x_1 x_2^2]}{2\pi(\lambda^* + 2\mu)x^4} \\ = & \frac{-2\mu F_2 x_2 + 2\mu F_1 x_1}{2\pi(\lambda^* + 2\mu)x^2} + \frac{2(\mu + \lambda^*)[x_1^2(F_1 x_1 + F_2 x_2) - x_2^2(F_2 x_2 + F_1 x_1)]}{2\pi(\lambda^* + 2\mu)x^4} \end{aligned}$$

$$= \frac{-\mu F_2 x_2 + \mu F_1 x_1}{\pi(\lambda^* + 2\mu)x^2} + \frac{(\mu + \lambda^*)(F_1 x_1 + F_2 x_2)(x_1^2 - x_2^2)}{\pi(\lambda^* + 2\mu)x^4}. \quad (4.32)$$

Notice that from equation (2.114),  $\sigma_{12}$  can be calculated as

$$\sigma_{12} = \frac{-\mu F_1 x_2}{2\pi(\lambda^* + 2\mu)x^2} - \frac{\mu F_2 x_1}{2\pi(\lambda^* + 2\mu)x^2} + \frac{\mu \delta_{12} F_\gamma x_\gamma}{2\pi(\lambda^* + 2\mu)x^2} - \frac{2\mu(\lambda^* + \mu)[F_1 x_1^2 x_2 + F_2 x_2^2 x_1]}{2\pi\mu(\lambda^* + 2\mu)x^4}, \quad (4.33)$$

or equivalently,

$$\sigma_{12} = \frac{-\mu F_1 x_2 - \mu F_2 x_1}{2\pi(\lambda^* + 2\mu)x^2} - \frac{\mu(\lambda^* + \mu)[F_1 x_1^2 x_2 + F_2 x_2^2 x_1]}{\pi\mu(\lambda^* + 2\mu)x^4}. \quad (4.34)$$

Thus (4.32) and (4.34) yields

$$\begin{aligned} \Sigma = \sigma_{22} - \sigma_{11} + 2i\sigma_{12} = & \frac{-\mu F_2 x_2 + \mu F_1 x_1}{\pi(\lambda^* + 2\mu)x^2} + \frac{(\mu + \lambda^*)(F_1 x_1 + F_2 x_2)(x_1^2 - x_2^2)}{\pi(\lambda^* + 2\mu)x^4} + \\ & 2i \left\{ \frac{-\mu F_1 x_2 - \mu F_2 x_1}{2\pi(\lambda^* + 2\mu)x^2} - \frac{\mu(\lambda^* + \mu)[F_1 x_1^2 x_2 + F_2 x_2^2 x_1]}{\pi\mu(\lambda^* + 2\mu)x^4} \right\}, \end{aligned} \quad (4.35)$$

and equation (4.35) can be evaluated as

$$\Sigma = \sigma_{22} - \sigma_{11} + 2i\sigma_{12} = \frac{-\mu F_2 x_2 + \mu F_1 x_1 + i(-\mu F_1 x_2 - \mu F_2 x_1)}{\pi(\lambda^* + 2\mu)x^2} + \frac{(\mu + \lambda^*)\{(F_1 x_1 + F_2 x_2)(x_1^2 - x_2^2) - 2i(F_1 x_1^2 x_2 + F_2 x_2^2 x_1)\}}{\pi(\lambda^* + 2\mu)x^4}. \quad (4.36)$$

Simplifying equation (4.36) yields

$$\begin{aligned} \Sigma &= \frac{\mu[-F_2 x_2 + F_1 x_1 - iF_1 x_2 - iF_2 x_1]}{\pi(\lambda^* + 2\mu)x^2} + \frac{(\mu + \lambda^*)\{(F_1 x_1 + F_2 x_2)(x_1^2 - x_2^2) - 2i x_1 x_2 (F_1 x_1 + F_2 x_2)\}}{\pi(\lambda^* + 2\mu)x^4} \\ &= \frac{\mu[-F_2(x_2 + ix_1) + F_1(x_1 - ix_2)]}{\pi(\lambda^* + 2\mu)x^2} + \frac{(\mu + \lambda^*)\{(F_1 x_1 + F_2 x_2)(x_1^2 - x_2^2) - 2i x_1 x_2\}}{\pi(\lambda^* + 2\mu)x^4}. \end{aligned} \quad (4.37)$$

We can rewrite the equation (4.37) in terms of  $z$  as

$$\Sigma = \frac{\mu}{\pi(\lambda^* + 2\mu)} \frac{[-F_2(i\bar{z}) + F_1(\bar{z})]}{|z|^2} + \frac{(\mu + \lambda^*)}{\pi(\lambda^* + 2\mu)} \frac{\operatorname{Re}\{F\bar{z}\}(\bar{z})^2}{|z|^4}. \quad (4.38)$$

Factoring  $|z|^2 = z\bar{z}$  in (4.38), we obtain

$$\begin{aligned} \Sigma &= \frac{\mu}{\pi(\lambda^* + 2\mu)} \frac{\bar{z}(F_1 - iF_2)}{|z|^2} + \frac{(\mu + \lambda^*)}{\pi(\lambda^* + 2\mu)} \frac{\operatorname{Re}\{F\bar{z}\}(\bar{z})^2}{|z|^4} \\ &= \frac{\mu}{\pi(\lambda^* + 2\mu)} \frac{\bar{z}\bar{F}}{z\bar{z}} + \frac{(\mu + \lambda^*)}{\pi(\lambda^* + 2\mu)} \operatorname{Re}\{F\bar{z}\} \frac{(\bar{z})^2}{(z\bar{z})^2} \\ &= \frac{\mu}{\pi(\lambda^* + 2\mu)} \frac{\bar{F}}{z} + \frac{(\mu + \lambda^*)}{\pi(\lambda^* + 2\mu)} \operatorname{Re}\{F\bar{z}\} \frac{1}{z^2}, \end{aligned} \quad (4.39)$$

which simplifies to

$$\begin{aligned}
 \Sigma &= \frac{\mu}{\pi(\lambda^* + 2\mu)} \frac{\bar{F}}{z} + \frac{(\mu + \lambda^*)}{\pi(\lambda^* + 2\mu)} \frac{F\bar{z} + \bar{F}z}{2z^2} \\
 &= \frac{2\mu z \bar{F} + (\mu + \lambda^*) F\bar{z} + (\mu + \lambda^*) \bar{F}z}{2\pi(\lambda^* + 2\mu) z^2} \\
 &= \frac{(3\mu + \lambda^*) z \bar{F} + (\mu + \lambda^*) F\bar{z}}{2\pi(\lambda^* + 2\mu) z^2}.
 \end{aligned} \tag{4.40}$$

Since equation (4.5),  $\Sigma = \sigma_{22} - \sigma_{11} + 2i\sigma_{12} = [2\bar{z}\phi''(z) + 2\psi'(z)]$ , then it follows from (4.40)

$$\frac{(\mu + \lambda^*)}{2\pi(\lambda^* + 2\mu)} \bar{z} \frac{F}{z^2} + \frac{(3\mu + \lambda^*)}{2\pi(\lambda^* + 2\mu)} \frac{\bar{F}}{z} = [2\bar{z}\phi''(z) + 2\psi'(z)]. \tag{4.41}$$

This leads to

$$\psi'(z) = \frac{(3\mu + \lambda^*)}{4\pi(\lambda^* + 2\mu)} \frac{\bar{F}}{z}, \tag{4.42}$$

and

$$\phi''(z) = \frac{(\mu + \lambda^*)}{4\pi(\lambda^* + 2\mu)} \frac{F}{z^2}. \tag{4.43}$$

Equation (4.43) after integrating twice gives

$$\phi(z) = \frac{-(\lambda^* + \mu)}{4\pi(\lambda^* + 2\mu)} F \ln(z), \quad (4.44)$$

and integrates (4.42) once gives

$$\psi(z) = \frac{(3\mu + \lambda^*)}{4\pi(\lambda^* + 2\mu)} \bar{F} \ln z, \quad (4.45)$$

or equivalently,

$$\psi(z) = \frac{(3-4\nu)\bar{F} \ln z}{8\pi(1-\nu)}. \quad (4.46)$$

#### 4.2.2.2 The complex potential representation of the components of displacements induced by point forces

One can know the most general expressions for the complex stress functions  $\phi$  and  $\psi$  in terms of the components of displacements induced by a point force as follows:

From (2.106) we can find the displacements induced by a point force,

$$u_1 = \frac{\lambda^* + 3\mu}{4(\lambda^* + 2\mu)\pi\mu} \delta_{1\gamma} F_\gamma \ln \frac{1}{x} + \frac{\lambda^* + \mu}{4\pi\mu(\lambda^* + 2\mu)} \frac{x_1 x_\gamma}{x^2} F_\gamma$$



$$\begin{aligned}
&= \frac{\lambda^* + 3\mu}{4(\lambda^* + 2\mu)\pi\mu} [\delta_{11} F_1 \ln \frac{1}{x} + \delta_{12} F_2 \ln \frac{1}{x}] + \frac{\lambda^* + \mu}{4\pi\mu(\lambda^* + 2\mu)} \left[ \frac{x_1 x_1}{x^2} F_1 + \frac{x_1 x_2}{x^2} F_2 \right] \\
&= \frac{\lambda^* + 3\mu}{4(\lambda^* + 2\mu)\pi\mu} F_1 \ln \frac{1}{x} + \frac{\lambda^* + \mu}{4\pi\mu(\lambda^* + 2\mu)} \frac{(x_1^2 F_1 + x_1 x_2 F_2)}{x^2} \\
&= \frac{\lambda^* + 3\mu}{4(\lambda^* + 2\mu)\pi\mu} F_1 \ln \frac{1}{x} + \frac{\lambda^* + \mu}{4\pi\mu(\lambda^* + 2\mu)} \frac{x_1 (x_1 F_1 + x_2 F_2)}{x^2}. \tag{4.47}
\end{aligned}$$

Therefore we can write (4.47) as

$$u_1 = \frac{\lambda^* + 3\mu}{4(\lambda^* + 2\mu)\pi\mu} F_1 \ln \frac{1}{|z|} + \frac{\lambda^* + \mu}{4\pi\mu(\lambda^* + 2\mu)} \frac{x_1 \operatorname{Re}\{F\bar{z}\}}{|z|^2}, \tag{4.48}$$

where  $\operatorname{Re}\{F\bar{z}\} = x_1 F_1 + x_2 F_2$  and  $|z| = x$ . Similarly,

$$u_2 = \frac{\lambda^* + 3\mu}{4(\lambda^* + 2\mu)\pi\mu} F_2 \ln \frac{1}{|z|} + \frac{\lambda^* + \mu}{4\pi\mu(\lambda^* + 2\mu)} \frac{x_2 \operatorname{Re}\{F\bar{z}\}}{|z|^2}. \tag{4.49}$$

By using (4.48) and (4.49) one can calculate

$$2\mu(u_1 - iu_2) = \frac{\lambda^* + 3\mu}{2(\lambda^* + 2\mu)\pi} (F_1 - iF_2) \ln \frac{1}{|z|} + \frac{\lambda^* + \mu}{2\pi(\lambda^* + 2\mu)} \frac{\operatorname{Re}\{F\bar{z}\}(x_1 - ix_2)}{|z|^2}. \tag{4.50}$$

Therefore,

$$2\mu(u_1 - iu_2) = \frac{\lambda^* + 3\mu}{2(\lambda^* + 2\mu)\pi} \bar{F} \ln \frac{1}{|z|} + \frac{\lambda^* + \mu}{2\pi(\lambda^* + 2\mu)} \operatorname{Re} \left\{ \frac{F \bar{z}}{|z|^2} \right\} \bar{z}, \quad (4.51)$$

where  $\bar{F} = F_1 - iF_2$  and  $\bar{z} = x_1 - ix_2$ . Thus by using the standard properties of the logarithm, and noting that  $\operatorname{Re} \left\{ \frac{F}{z} \right\} = \frac{1}{2} \left( \frac{F}{z} + \frac{\bar{F}}{\bar{z}} \right)$ , we can simplify equation (4.51) to

$$\begin{aligned} 2\mu(u_1 - iu_2) &= \frac{\lambda^* + 3\mu}{2(\lambda^* + 2\mu)\pi} (-\bar{F} \ln |z|) + \frac{\lambda^* + \mu}{2\pi(\lambda^* + 2\mu)} \operatorname{Re} \left\{ \frac{F \bar{z}}{(z \bar{z})} \right\} \bar{z} \\ &= \frac{\lambda^* + 3\mu}{2(\lambda^* + 2\mu)\pi} (-\bar{F} \ln(z \bar{z})^{\frac{1}{2}}) + \frac{\lambda^* + \mu}{2\pi(\lambda^* + 2\mu)} \operatorname{Re} \left\{ \frac{F}{z} \right\} \bar{z} \\ &= \frac{\lambda^* + 3\mu}{2(\lambda^* + 2\mu)\pi} \left[ -\frac{1}{2} \bar{F} \ln z - \frac{1}{2} \bar{F} \ln \bar{z} \right] + \frac{\lambda^* + \mu}{2\pi(\lambda^* + 2\mu)} \frac{1}{2} \left( \frac{F}{z} + \frac{\bar{F}}{\bar{z}} \right) \bar{z} \\ &= \frac{-(\lambda^* + 3\mu)}{4(\lambda^* + 2\mu)\pi} [\bar{F} \ln z + \bar{F} \ln \bar{z}] + \frac{\lambda^* + \mu}{4\pi(\lambda^* + 2\mu)} \left( \frac{F}{z} \bar{z} + \bar{F} \right). \end{aligned} \quad (4.52)$$

Since the equation (4.21),  $2\mu(u_1 - iu_2) = -\bar{z}\phi' + k\bar{\phi} - \psi$ , then it follows from (4.52)

$$-\bar{z}\phi' + k\bar{\phi} - \psi = \bar{z} \frac{(\lambda^* + \mu)}{4\pi(\lambda^* + 2\mu)} \frac{F}{z} + k \frac{-(\lambda^* + \mu)}{4\pi(\lambda^* + 2\mu)} \bar{F} \ln(\bar{z}) - \frac{(\lambda^* + 3\mu)}{4\pi(\lambda^* + 2\mu)} \bar{F} \ln(z). \quad (4.53)$$

Therefore,

$$\phi'(z) = \frac{-(\lambda^* + \mu)}{4\pi(\lambda^* + 2\mu)} \frac{F}{z}, \quad \bar{\phi}(z) = \frac{-(\lambda^* + \mu)}{4\pi(\lambda^* + 2\mu)} \bar{F} \ln(\bar{z}),$$

and

$$\psi(z) = \frac{(\lambda^* + 3\mu)}{4\pi(\lambda^* + 2\mu)} \bar{F} \ln(z) .$$

The following section will present a classical representation of the Westergaard's method for developing the stress functions near to the crack tip.

### 4.3 Westergaard method for mode I (symmetric) problems

We consider an infinite plane with a crack along the  $x_1$ -axis. We assume that the external loads are symmetric with respect to the  $x_1$ -axis, then  $\sigma_{12}=0$  along  $x_2=0$ .

Let

$$\psi' = -z\varphi'' . \quad (4.54)$$

Substituting (4.54) into (4.5) and solving the resulting equation, we get

$$\begin{aligned} \sigma_{22} - \sigma_{11} &= \operatorname{Re}\{2(\bar{z}\varphi'' - z\varphi'')\} = \operatorname{Re}\{2\varphi''(\bar{z} - z)\} \\ &= \operatorname{Re}\{2\varphi''(2i)\frac{(\bar{z} - z)}{2i}\} = \operatorname{Re}\{-4i\varphi''\frac{(z - \bar{z})}{2i}\} \\ &= \operatorname{Re}\{-4i\varphi''\operatorname{Im}(z)\} = \operatorname{Re}\{-4i\varphi''x_2\} \\ &= 4x_2\operatorname{Im}\{\varphi''\}, \end{aligned}$$

where

$$\operatorname{Re}\{i\varphi''(z)x_2\} = \operatorname{Re}\{ix_2[\varphi_1''(z) + i\varphi_2''(z)]\} = \operatorname{Re}\{ix_2\varphi_1''(z) - x_2\varphi_2''(z)\} = -x_2\varphi_2''(z) = -x_2\operatorname{Im}\{\varphi''(z)\} .$$

The  $\sigma_{22}$  stress can therefore be written as

$$\begin{aligned} 2\sigma_{22} &= 4\operatorname{Re}\{\varphi'\} + \operatorname{Re}\{2(\bar{z}\varphi'' + \psi'')\} \\ &= 4\operatorname{Re}\{\varphi'\} + 4x_2\operatorname{Im}\{\varphi''\}, \end{aligned}$$

and thus,

$$\sigma_{22} = 2\{\operatorname{Re}\{\varphi'\} + x_2\operatorname{Im}\{\varphi''\}\}.$$

From (4.4) we have

$$\begin{aligned} \sigma_{11} &= 4\operatorname{Re}\{\varphi'\} - \sigma_{22} = 4\operatorname{Re}\{\varphi'\} - 2\{\operatorname{Re}\{\varphi'\} + x_2\operatorname{Im}\{\varphi''\}\} \\ &= 2\{\operatorname{Re}\{\varphi'\} - x_2\operatorname{Im}\{\varphi''\}\}. \end{aligned}$$

From (4.5) we have

$$\sigma_{12} = \operatorname{Im}\{\bar{z}\varphi'' + \psi''\} = -2x_2\operatorname{Re}\{\varphi''\}.$$

Thus the stresses can be written as

$$\begin{aligned} \sigma_{11} &= 2\{\operatorname{Re}\{\varphi'\} - x_2\operatorname{Im}\{\varphi''\}\}, \\ \sigma_{22} &= 2\{\operatorname{Re}\{\varphi'\} + x_2\operatorname{Im}\{\varphi''\}\}, \\ \sigma_{12} &= -2x_2\operatorname{Re}\{\varphi''\}. \end{aligned} \tag{4.55}$$

The displacements are

$$\begin{aligned} 2\mu u_1 &= \frac{k-1}{2} \operatorname{Re} \varphi - x_2 \operatorname{Im} \varphi', \\ 2\mu u_2 &= \frac{k+1}{2} \operatorname{Im} \varphi - x_2 \operatorname{Re} \varphi'. \end{aligned} \quad (4.56)$$

We define  $Z$  as a Westergaard function for Mode I problems by

$$\varphi' = \frac{1}{2} Z. \quad (4.57)$$

Therefore,

$$\varphi = \frac{1}{2} \hat{Z}, \quad \varphi'' = \frac{1}{2} Z'. \quad (4.58)$$

Using these equations in (4.55) and (4.56) we get the stresses that were proposed by Westergaard as the stress singularity field at the crack tip as

$$\begin{aligned} \sigma_{11} &= \operatorname{Re} Z - x_2 \operatorname{Im} Z', \\ \sigma_{22} &= \operatorname{Re} Z + x_2 \operatorname{Im} Z', \\ \sigma_{12} &= -x_2 \operatorname{Re} Z'. \end{aligned} \quad (4.59)$$

#### 4.4 Westergaard method for Mode II (sliding mode) problems

Consider an infinite plane with cracks along the  $x_1$ -axis and the external loads are anti-symmetric with respect to the  $x_1$ -axis, then  $\sigma_{22}=0$  along  $x_2=0$ . Let

$$\psi' = -2\varphi' - z\varphi''. \quad (4.60)$$

Then the method outlined in section 4.2.1 can be used to calculate the stresses and the displacements. The stresses are

$$\begin{aligned} \sigma_{11} &= 2\{\operatorname{Re}\{\varphi'\} - x_2 \operatorname{Im}\{\varphi''\}\}, \\ \sigma_{12} &= 2\{-\operatorname{Im}\{\varphi'\} - x_2 \operatorname{Re}\{\varphi''\}\}, \\ \sigma_{22} &= 2x_2 \operatorname{Re}\{\varphi''\}. \end{aligned} \quad (4.61)$$

The displacements are

$$\begin{aligned} 2\mu u_1 &= \frac{k+1}{2} \operatorname{Re}\varphi - x_2 \operatorname{Im}\varphi', \\ 2\mu u_2 &= \frac{k-1}{2} \operatorname{Im}\varphi - x_2 \operatorname{Re}\varphi'. \end{aligned} \quad (4.62)$$

We define  $Z$  as a Westergaard function for Mode II problems by

$$Z_{II} = 2i\psi'(z). \quad (4.63)$$

By using (4.59) one can get,

$$\begin{aligned}
 \sigma_{11} &= 2\operatorname{Re}Z - x_2 \operatorname{Im}Z', \\
 \sigma_{12} &= -\operatorname{Im}Z - x_2 \operatorname{Re}Z', \\
 \sigma_{22} &= x_2 \operatorname{Im}Z'.
 \end{aligned}
 \tag{4.64}$$

#### 4.5 The elasticity solutions by using the Westergaard function method

Consider a crack of length  $2a$  in an infinite plate subjected to compressive forces  $P$  and  $Q$  at  $x=b$  and assumed  $Q$  to be absent in this case.

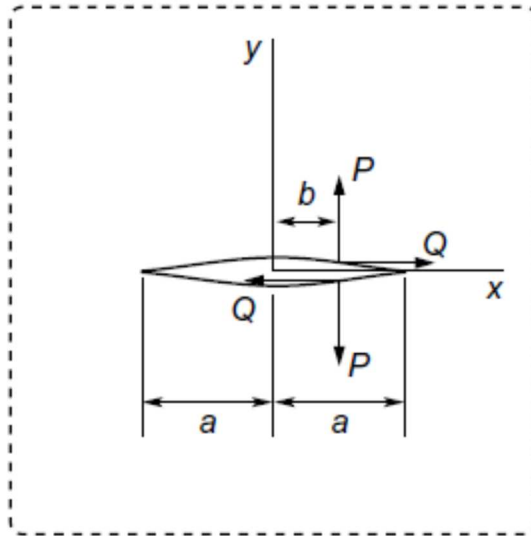


Figure 4.1 A crack in an infinite plate subjected to compressive forces [54]

Let  $Z$  be a Westergaard function defined as

$$Z = \frac{P}{\pi(z-t)} \sqrt{\frac{a^2 - t^2}{z^2 - a^2}}.
 \tag{4.65}$$

Then

$$\begin{aligned} Z &= \frac{P}{\pi((t+w)-t)} \sqrt{\frac{a^2-t^2}{(t+w)^2-a^2}} \\ &= \frac{P}{\pi w} \sqrt{\frac{a^2-t^2}{(t+w)^2-a^2}} \end{aligned}$$

Where  $z=t+w, |w| \ll 1$ . Since  $w$  is too small, then can be considered zero. Therefore,

$$\sqrt{\frac{a^2-t^2}{t^2-a^2}} = \sqrt{\frac{-(t^2-a^2)}{t^2-a^2}} = \sqrt{-1} = \pm i$$

It follows that

$$Z = \frac{-iP}{\pi w}$$

Define



$$\begin{aligned}
 \phi' &= \frac{1}{2} Z \\
 &= \frac{1}{2} \left( \frac{-i P}{\pi w} \right) \\
 &= \frac{1}{2} \left( \frac{-i P}{\pi(z-t)} \right).
 \end{aligned}$$

Now, we would like to show that the Westergaard function gives the stresses that satisfy the following boundary conditions.

Define analytic function  $\psi''$  as

$$\psi''(z) = -z\phi''(z). \quad (4.66)$$

Then

$$\phi''(z) = \frac{1}{2} \left[ \frac{-Pi}{\pi} \cdot \frac{-1}{(z-t)^2} \right] = \frac{i P}{2\pi(z-t)^2}.$$

We can write  $\psi''$  as

$$\psi''(z) = -z\phi''(z) = \frac{-i P z}{2\pi(z-t)^2}.$$

Let  $w=z-t$ . Then

$$Z = \frac{-iP}{\pi w} \Rightarrow Z = \frac{-iP(x-iy)}{\pi |w|^2} = \frac{-iPx - Py}{\pi(x^2 + y^2)},$$

and

$$\operatorname{Re}\{Z\} = \frac{-Py}{\pi(x^2 + y^2)} = \frac{-P\varepsilon}{\pi(x^2 + \varepsilon^2)}.$$

Now,

$$\begin{aligned} \int_{-\delta}^{\delta} \frac{-P\varepsilon}{\pi(x^2 + \varepsilon^2)} dx &= \frac{-P\varepsilon}{\pi} \int_{-\delta}^{\delta} \frac{dx}{(x^2 + \varepsilon^2)} = \frac{-\varepsilon P}{\pi} \left( \frac{1}{\varepsilon} \tan^{-1}\left(\frac{x}{\varepsilon}\right) \right) \Big|_{-\delta}^{\delta} = \frac{-P}{\pi} \tan^{-1}\left(\frac{\delta}{\varepsilon}\right) - \frac{P}{\pi} \tan^{-1}\left(\frac{-\delta}{\varepsilon}\right) \\ &\xrightarrow{\varepsilon \rightarrow 0} \frac{P}{\pi} \tan^{-1}(\infty) - \frac{P}{\pi} \tan^{-1}(-\infty) \\ &= \frac{P}{\pi} \left[ \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \right] \\ &= P. \end{aligned}$$

Since

$$\begin{aligned} \sigma_{11} &= \operatorname{Re}Z - y \operatorname{Im}Z', \\ \sigma_{22} &= \operatorname{Re}Z + y \operatorname{Im}Z', \\ \sigma_{12} &= -y \operatorname{Re}Z', \end{aligned}$$

and

$$Z = \frac{-ip}{\pi z} \Rightarrow Z' = \frac{ip}{\pi z^2},$$

$$Z = \frac{-ip}{\pi z} \cdot \frac{\bar{z}}{\bar{z}} = \frac{-ip\bar{z}}{\pi|z|^2} = \frac{-ip(x-iy)}{\pi(x^2+y^2)} = \frac{-ipx-py}{\pi(x^2+y^2)},$$

then

$$\operatorname{Re}\{Z\} = \frac{-py}{\pi(x^2+y^2)}, \operatorname{Im}\{Z\} = \frac{-px}{\pi(x^2+y^2)}.$$

$$\begin{aligned} Z' &= \frac{ip}{\pi z^2} = \frac{ip}{\pi z^2} \cdot \frac{\bar{z}^2}{\bar{z}^2} = \frac{ip\bar{z}^2}{\pi|z|^4} = \frac{-ip(x-iy)^2}{\pi(x^2+y^2)^2} \\ &= \frac{ip[x^2-2ixy-y^2]}{\pi(x^2+y^2)^2} \\ &= \frac{ipx^2+2xyp-ipy^2}{\pi(x^2+y^2)^2}. \end{aligned}$$

Therefore,

$$\operatorname{Re}\{Z'\} = \frac{2xyp}{\pi(x^2+y^2)^2}, \operatorname{Im}\{Z'\} = \frac{x^2p-py^2}{\pi(x^2+y^2)^2}.$$

Since

$$\begin{aligned}\sigma_{12} &= -y \operatorname{Re}\{Z'\} = -y \left( \frac{2xyP}{\pi(x^2+y^2)^2} \right) \\ &= \frac{-2xy^2P}{\pi(x^2+y^2)^2},\end{aligned}$$

then

$$\int_{-\delta}^{\delta} \sigma_{12} dx = \int_{-\delta}^{\delta} \frac{-2xy^2P}{\pi(x^2+y^2)^2} dx = \frac{-2Py^2}{\pi} \int_{-\delta}^{\delta} \frac{x}{(x^2+y^2)^2} dx.$$

Let

$$\begin{aligned}u &= (x^2+y^2)^2 \Rightarrow du = 2(x^2+y^2)(2x)dx \\ u^{\frac{1}{2}} &= (x^2+y^2) \Rightarrow du = 2u^{\frac{1}{2}}(2x)dx \Rightarrow \frac{du}{4u^{1/2}} = x dx.\end{aligned}$$

Then

$$\begin{aligned}
 \int_{-\delta}^{\delta} \frac{x \, dx}{(x^2+y^2)^2} &= \frac{1}{4} \int_{-\delta}^{\delta} \frac{u^{-1/2} du}{u} = \frac{1}{4} \int_{-\delta}^{\delta} \frac{du}{u^{1+\frac{1}{2}}} = \frac{1}{4} \int_{-\delta}^{\delta} \frac{du}{u^{\frac{3}{2}}} = \frac{1}{4} \int_{-\delta}^{\delta} u^{-3/2} du \\
 &= (-2) \frac{1}{4} u^{\frac{-1}{2}} \Big|_{-\delta}^{\delta} = \frac{-1}{2} (x^2+y^2)^{-1} \Big|_{-\delta}^{\delta} = \frac{-1}{2(\delta^2+y^2)} + \frac{1}{2(\delta^2+y^2)} = 0.
 \end{aligned}$$

Therefore,

$$\int_{-\delta}^{\delta} \sigma_{12} \, dx = \int_{-\delta}^{\delta} \frac{-2xy^2P}{\pi(x^2+y^2)^2} \, dx = \frac{-2Py^2}{\pi} \int_{-\delta}^{\delta} \frac{x}{(x^2+y^2)^2} \, dx = 0.$$

Since

$$\begin{aligned}
 \sigma_{22} &= \operatorname{Re} Z + y \operatorname{Im} Z' \\
 &= \frac{-Py}{\pi(x^2+y^2)} + y \left[ \frac{x^2P - Py^2}{\pi(x^2+y^2)^2} \right] \\
 &= \frac{-Py(x^2+y^2)}{\pi(x^2+y^2)^2} + \frac{yP(x^2-y^2)}{\pi(x^2+y^2)^2} \\
 &= \frac{-2Py^3}{\pi(x^2+y^2)^2},
 \end{aligned}$$

then

$$\int_{-\delta}^{\delta} \sigma_{22} \, dx = \int_{-\delta}^{\delta} \frac{-2Py^3}{\pi(x^2+y^2)^2} \, dx = \frac{-2Py^3}{\pi} \int_{-\delta}^{\delta} \frac{dx}{(x^2+y^2)^2}.$$

This integral  $\int_{-\delta}^{\delta} \frac{dx}{(x^2+y^2)^2}$  can be done by substitution

$$x=y \tan \theta \Rightarrow dx=y \sec^2 \theta d\theta$$

$$x^2+y^2=y^2 \tan^2 \theta+y^2=y^2 (\tan^2 \theta+1)=y^2 \sec^2 \theta$$

$$\frac{x}{y} = \tan \theta \Rightarrow \theta = \tan^{-1} \left( \frac{x}{y} \right).$$

Now,

$$\begin{aligned} \int_{-\delta}^{\delta} \frac{dx}{(x^2+y^2)^2} &= \int \frac{y \sec^2 \theta d\theta}{(y^2 \sec^2 \theta)^2} = \int \frac{d\theta}{y^3 \sec^2 \theta} \\ &= \frac{1}{y^3} \int \frac{d\theta}{\sec^2 \theta} = \frac{1}{y^3} \int \frac{d\theta}{\frac{1}{\cos^2 \theta}} \\ &= \frac{1}{y^3} \int \cos^2 \theta d\theta = \frac{1}{y^3} \int \frac{(1+\cos 2\theta)}{2} d\theta = \frac{1}{2y^3} \int (1+\cos 2\theta) d\theta. \end{aligned}$$

Let

$$u=2\theta \Rightarrow du=2d\theta \Rightarrow d\theta = \frac{du}{2}.$$

Then

$$\int \cos 2\theta \, d\theta = \frac{1}{2} \int \cos u \, du = \frac{1}{2} \sin u = \frac{1}{2} \sin 2\theta ,$$

and

$$\frac{1}{2y^3} \int (1 + \cos 2\theta) \, d\theta = \frac{1}{2y^3} \left[ \theta + \frac{1}{2} \sin 2\theta \right] .$$

Therefore,

$$\begin{aligned} \int_{-\delta}^{\delta} \frac{dx}{(x^2 + y^2)^2} &= \frac{1}{2y^3} \left[ \theta + \frac{1}{2} \sin 2\theta \right] \bigg|_{\tan^{-1}(\frac{-\delta}{y})}^{\tan^{-1}(\frac{\delta}{y})} \\ &= \frac{1}{2y^3} \left\{ \left[ \tan^{-1}\left(\frac{\delta}{y}\right) - \tan^{-1}\left(\frac{-\delta}{y}\right) \right] + \frac{1}{2} \sin 2\left(\tan^{-1}\left(\frac{\delta}{y}\right)\right) - \frac{1}{2} \sin 2\left(\tan^{-1}\left(\frac{-\delta}{y}\right)\right) \right\} \\ &= \frac{1}{2y^3} \left\{ \left[ \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \right] + \frac{1}{2} \sin 2\left(\frac{\pi}{2}\right) - \frac{1}{2} \sin 2\left(\frac{-\pi}{2}\right) \right\} \\ &= \frac{\pi}{2y^3} . \end{aligned}$$

$$\int_{-\delta}^{\delta} \sigma_{22} \, dx = \frac{-2Py^3}{\pi} \int_{-\delta}^{\delta} \frac{dx}{(x^2 + y^2)^2} = \left( \frac{-2Py^3}{\pi} \right) \left( \frac{\pi}{2y^3} \right) = -P .$$

Now,

$$\begin{aligned}
 \sigma_{11} &= \operatorname{Re} Z - y \operatorname{Im} Z' \\
 &= \frac{-Py}{\pi(x^2+y^2)} - y \left[ \frac{x^2 P - Py^2}{\pi(x^2+y^2)^2} \right] \\
 &= \frac{-Py(x^2+y^2)}{\pi(x^2+y^2)^2} - \frac{yP(x^2-y^2)}{\pi(x^2+y^2)^2} \\
 &= \frac{-2Pyx^2}{\pi(x^2+y^2)^2}.
 \end{aligned}$$

One can find the integral  $\int_{-\delta}^{\delta} \sigma_{11} dx$  as

$$\int_{-\delta}^{\delta} \sigma_{11} dx = \int_{-\delta}^{\delta} \frac{-2Pyx^2}{\pi(x^2+y^2)^2} dx = \frac{-2Py}{\pi} \int_{-\delta}^{\delta} \frac{x^2 dx}{(x^2+y^2)^2}.$$

This integral  $\int_{-\delta}^{\delta} \frac{x^2 dx}{(x^2+y^2)^2}$  can be done by parts with

$$u=x, dv=\frac{xdx}{(x^2+y^2)^2} \Rightarrow du=dx, v=\frac{-1}{2(x^2+y^2)}.$$

Then

$$\int_{-\delta}^{\delta} \frac{x^2 dx}{(x^2+y^2)^2} = \frac{-x}{2(x^2+y^2)} \Big|_{-\delta}^{\delta} + \int_{-\delta}^{\delta} \frac{dx}{2(x^2+y^2)},$$



and

$$\frac{1}{2} \int_{-\delta}^{\delta} \frac{dx}{(x^2+y^2)} = \frac{1}{2} \left( \frac{1}{y} \tan^{-1} \left( \frac{x}{y} \right) \right) \Big|_{-\delta}^{\delta} = \frac{1}{2y} \left[ \frac{\pi}{2} - \left( -\frac{\pi}{2} \right) \right] = \frac{\pi}{2y}.$$

Hence,

$$\begin{aligned} \int_{-\delta}^{\delta} \frac{x^2 dx}{(x^2+y^2)^2} &= \frac{-x}{2(x^2+y^2)} \Big|_{-\delta}^{\delta} + \int_{-\delta}^{\delta} \frac{dx}{2(x^2+y^2)} \\ &= \frac{-\delta}{2(\delta^2+y^2)} + \frac{-\delta}{2(\delta^2+y^2)} + \frac{\pi}{2y}. \end{aligned}$$

$$\int_{-\delta}^{\delta} \sigma_{11} dx = \int_{-\delta}^{\delta} \frac{-2Pyx^2}{\pi(x^2+y^2)^2} dx = \frac{-2Py}{\pi} \int_{-\delta}^{\delta} \frac{x^2 dx}{(x^2+y^2)^2} = \frac{-2Py}{\pi} \left[ \frac{-2\delta}{2(\delta^2+y^2)} + \frac{\pi}{2y} \right] = -P, y=\epsilon \rightarrow 0.$$

## 4.6 A new Westergaard's function to model the shear stresses along the crack flanks

We note here the following version of Westergaard's function which could be used for future work. This function models the shear stresses along the crack flanks predicted by the CJP model.

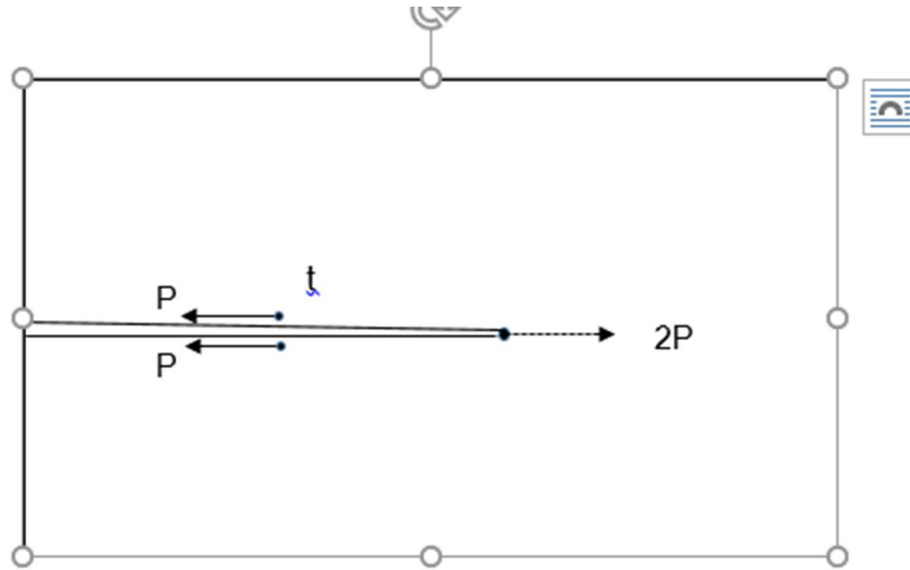


Figure 4.2 A new Westergaard's function to model the shear stresses along the crack flanks

We defined a new function  $Z$  as

$$Z = \frac{P\sqrt{a^2-t^2}}{\pi(z-t)\sqrt{z^2-a^2}} \ln \left[ \frac{(z-a)(t+a)}{(z+a)(a-t)} \right]. \quad (4.67)$$

By evaluating this function as

$$\begin{aligned} Z &= \frac{P\sqrt{a^2-t^2}}{\pi(z-t)\sqrt{z^2-a^2}} \ln \left[ \frac{(z-a)(t+a)}{(z+a)(a-t)} \right] \\ &= \frac{P}{\pi(z-t)} \sqrt{\frac{a^2-t^2}{z^2-a^2}} [\ln(z-a) + \ln(t+a) - \ln(z+a) - \ln(a-t)] \end{aligned} \quad (4.68)$$

Equation (4.67) models not only the square root singularity at the crack tip but also the singularity associated with the point load located at  $y=0$ ,  $x=-a$ ,  $a$ .

Let  $z=-a+w \Rightarrow z+a=w$  and  $t-a=t^* \Rightarrow t=t^*+a$ . Then

$$Z = \frac{iP}{\pi(w-2a-t^*)} \sqrt{\frac{t^*(t^*+2a)}{w(w-2a)}} [\ln(w-2a) + \ln(t^*+2a) - \ln w - \ln(-t^*)]$$

Now,

$$f(w) = \ln(w-2a) = f(0) + \frac{f'(0)}{1!} (w) + \frac{f''(0)}{2!} (w^2) + \dots$$

$$= \ln(-2a) + \frac{-1}{2a} w - \frac{1}{8a^2} w^2 + \dots,$$

and

$$f(t^*) = \ln(t^*+2a) = f(0) + \frac{f'(0)}{1!} (t^*) + \frac{f''(0)}{2!} (t^{*2}) + \dots$$

$$= \ln(2a) + \frac{1}{2a} t^* - \frac{1}{8a^2} t^{*2} + \dots,$$

then

$$Z = \frac{iP}{\pi(w-2a-t^*)} \sqrt{\frac{t^*(t^*+2a)}{w(w-2a)}} [\ln(w-2a) + \ln(t^*+2a) - \ln w - \ln(-t^*)]$$

$$= \frac{iP}{\pi(w-2a-t^*)} \sqrt{\frac{t^*(t^*+2a)}{w(w-2a)}} [\ln(-2a) + \ln(2a) - \ln w - \ln(-t^*)]$$

Now, let a coordinate transformation on the left  $z=a+w \rightarrow z-a=w$  and  $t-a=t^* \rightarrow t=t^*+a$ .

Then from (4.68) we can get

$$\begin{aligned} Z &= \frac{iP}{\pi(w-t^*)} \sqrt{\frac{t^*(t^*+2a)}{w(w+2a)}} [\ln(w) + \ln(2a) - \ln(2a) - \ln(-t^*)] \\ &= \frac{iP}{\pi(w-t^*)} \sqrt{\frac{t^*(t^*+2a)}{w(w+2a)}} [\ln(w) - \ln(-t^*)] \\ &= \frac{iP}{\pi(w-t^*)} \sqrt{\frac{t^*(t^*+2a)}{w(w+2a)}} \ln\left(\frac{-w}{t^*}\right). \end{aligned}$$

Let  $z=t+w \rightarrow z-t=w$  and  $t-a=t^* \rightarrow t=t^*+a$ . Then from (4.67) one can write

$$\begin{aligned} Z &= \frac{iP}{\pi(w)} \frac{\sqrt{t^*(t^*+2a)}}{\sqrt{(w+t)^2 - a^2}} [\ln(t+w-a) + \ln(t+a) - \ln(t+w+a) - \ln(a-t)] \\ &= \frac{iP}{\pi(w)} \frac{\sqrt{t^*(t^*+2a)}}{\sqrt{(w+t)^2 - a^2}} [\ln(t-a) + \ln(2a) - \ln(t+a) - \ln(a)] \\ &= \frac{iP}{\pi(w)} \frac{\sqrt{t^*(t^*+2a)}}{\sqrt{(w+t)^2 - a^2}} [\ln(-a) + \ln(2a) - \ln(2a) - \ln(a)] \\ &= \frac{iP}{\pi(w)} \frac{\sqrt{t^*(t^*+2a)}}{\sqrt{(w+t)^2 - a^2}} [\ln(-a) - \ln(a)] \\ &= \frac{iP}{\pi(w)} \frac{\sqrt{t^*(t^*+2a)}}{\sqrt{(w+t)^2 - a^2}} \ln(-1) = \frac{iP}{\pi(w)} \frac{\sqrt{t^*(t^*+2a)}}{\sqrt{(w+t)^2 - a^2}} (i\pi) = \frac{-P}{(w)} \frac{\sqrt{t^*(t^*+2a)}}{\sqrt{(w+t)^2 - a^2}}. \end{aligned}$$

$$\phi' = \frac{\sigma_{\infty}}{\pi\sqrt{z^2 - a^2}} \int_{-a}^a \frac{\sqrt{a^2 - t^2}}{z - t} dt$$

$$\text{Let } t = \frac{1}{u} \Rightarrow dt = -\frac{1}{u^2} du$$

Then

$$\begin{aligned} \int_{-a}^a \frac{\sqrt{a^2 - t^2}}{z - t} dt &= \oint \frac{\sqrt{a^2 - \frac{1}{u^2}}}{z - \frac{1}{u}} \left(-\frac{1}{u^2} du\right) = \oint \frac{\sqrt{a^2 u^2 - 1}}{\frac{uz - 1}{u}} \left(-\frac{1}{u^2} du\right) \\ &= \oint \frac{\sqrt{a^2 u^2 - 1}}{uz - 1} \left(-\frac{1}{u^2} du\right) = \oint \frac{\sqrt{a^2 u^2 - 1}}{1 - uz} \left(\frac{1}{u^2} du\right) \end{aligned}$$

Since

$$\oint_C f(z) dz = 2\pi i \sum_{j=1}^m \text{Res}_{z_j} f$$

where

$$\text{Res}_{z_0} f = \lim_{z \rightarrow z_0} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} [(z - z_0)^n f(z)]$$

$$\begin{aligned}
\text{Res}_{u=0} \left( \frac{\sqrt{a^2 u^2 - 1}}{u^2 (1 - uz)} \right) &= \lim_{u \rightarrow 0} \frac{1}{(2-1)!} \frac{d^{2-1}}{dz^{2-1}} \left[ (u-0)^2 \frac{\sqrt{a^2 u^2 - 1}}{u^2 (1 - uz)} \right] \\
&= \lim_{u \rightarrow 0} \frac{d}{dz} \left( \frac{\sqrt{a^2 u^2 - 1}}{(1 - uz)} \right) \\
&= \lim_{u \rightarrow 0} \left( \frac{(1 - uz) \frac{1}{2\sqrt{a^2 u^2 - 1}} (2a^2 u) - \sqrt{a^2 u^2 - 1} (-z)}{(1 - uz)^2} \right) \\
&= z\sqrt{-1} = zi
\end{aligned}$$

$$\begin{aligned}
\text{Res}_{u=\frac{1}{z}} \left( \frac{\sqrt{a^2 u^2 - 1}}{u^2 (1 - uz)} \right) &= \lim_{u \rightarrow \frac{1}{z}} \left[ \left( u - \frac{1}{z} \right) \frac{\sqrt{a^2 u^2 - 1}}{u^2 (1 - uz)} \right] = \lim_{u \rightarrow \frac{1}{z}} \left[ \left( \frac{uz - 1}{z} \right) \frac{\sqrt{a^2 u^2 - 1}}{-u^2 (uz - 1)} \right] \\
&= \lim_{u \rightarrow \frac{1}{z}} \left[ \left( \frac{1}{z} \right) \frac{\sqrt{a^2 u^2 - 1}}{-u^2} \right] = \left( \frac{1}{z} \right) \frac{\sqrt{a^2 \left( \frac{1}{z} \right)^2 - 1}}{-\left( \frac{1}{z} \right)^2} = -\sqrt{a^2 - z^2}
\end{aligned}$$

$$\oint_C f(z) dz = 2\pi i \sum_{j=1}^m \text{Res}_{z_j} f$$

$$\begin{aligned}
\int_{-a}^a \frac{\sqrt{a^2 - t^2}}{z - t} dt &= \oint \frac{\sqrt{a^2 u^2 - 1}}{1 - uz} \left( \frac{1}{u^2} du \right) = 2i\pi [zi - \sqrt{a^2 - z^2}] \\
&= -2\pi z - 2i\pi \sqrt{a^2 - z^2}
\end{aligned}$$

$$\begin{aligned}
\phi' &= \frac{\sigma_\infty}{\pi \sqrt{z^2 - a^2}} \int_{-a}^a \frac{\sqrt{a^2 - t^2}}{z - t} dt = \frac{\sigma_\infty}{\pi \sqrt{z^2 - a^2}} [-2\pi z + 2\pi \sqrt{z^2 - a^2}] \\
&= \frac{-2\sigma_\infty z}{\sqrt{z^2 - a^2}} + 2\sigma_\infty
\end{aligned}$$

$$Z = \frac{p}{w} \ln t$$

$$Z' = \frac{-p}{w^2} \ln t$$

$$Z = \frac{p}{w} \frac{\bar{w}}{\bar{w}} \ln t = \frac{(p \ln t)(x-iy)}{|w|^2} = \frac{px \ln t - ipy \ln t}{(x^2 + y^2)}$$

$$\operatorname{Re}\{Z\} = \frac{px \ln t}{x^2 + y^2}$$

$$\operatorname{Im}\{Z\} = \frac{-py \ln t}{x^2 + y^2}$$

$$\begin{aligned} Z' &= \frac{-p}{w^2} \frac{\bar{w}^2}{\bar{w}^2} \ln t = \frac{(-p \ln t)(x-iy)^2}{|w|^4} = \frac{-p \ln t [x^2 - 2ixy - y^2]}{(x^2 + y^2)^2} \\ &= \frac{-p \ln t [(x^2 - y^2) - 2ixy]}{(x^2 + y^2)^2} \end{aligned}$$

$$\operatorname{Re}\{Z'\} = \frac{-p(x^2 - y^2) \ln t}{(x^2 + y^2)^2}$$

$$\operatorname{Im}\{Z'\} = \frac{2pxy \ln t}{(x^2 + y^2)^2}$$

$$Z_1 = \frac{-ip}{\pi w} \ln(t+w)$$

$$\ln(t+w) = \ln\left[t\left(1 + \frac{w}{t}\right)\right] = \ln t + \ln\left(1 + \frac{w}{t}\right)$$

$$= \ln t + \sum_{n=0}^{\infty} \frac{(-1)^n w^{n+1}}{(n+1)t^{n+1}}$$

$$Z_1 = \frac{-ip}{\pi w} \left[ \ln t + \sum_{n=0}^{\infty} \frac{(-1)^n w^{n+1}}{(n+1)t^{n+1}} \right] = \frac{-ip}{\pi w} \ln t$$

$$Z_1' = \frac{-ip}{\pi} (-1) w^{-2} \ln t = \frac{ip \ln t}{\pi w^2},$$

$t$  is a constant.

$$\begin{aligned}
\sigma_{12} &= -y \operatorname{Re} Z_1' \\
&= -y \left[ \frac{(-p \ln t)(x^2 - y^2)}{(x^2 + y^2)^2} \right] \\
&= \frac{(p \ln t)(x^2 - y^3)}{(x^2 + y^2)^2} \\
\int_{-\delta}^{\delta} \sigma_{12} dx &= p \ln t \int_{-\delta}^{\delta} \frac{x^2 y - y^3}{(x^2 + y^2)^2} dx = p \left( \frac{-\pi}{2y^2} \right) \ln t = \frac{-p\pi}{2y^2} \ln t
\end{aligned}$$

$$\begin{aligned}
\sigma_{22} &= \operatorname{Re} Z + y \operatorname{Im} Z' \\
&= \frac{px \ln t}{x^2 + y^2} + y \frac{2pxy \ln t}{(x^2 + y^2)^2} \\
&= \frac{px(x^2 + y^2) \ln t + 2pxy^2 \ln t}{(x^2 + y^2)^2} \\
&= \frac{px^3 \ln t + 2pxy^2 \ln t + xpy^2 \ln t}{(x^2 + y^2)^2} \\
&= \frac{px^3 \ln t + 3pxy^2 \ln t}{(x^2 + y^2)^2}
\end{aligned}$$

$$\begin{aligned}
\int_{-\delta}^{\delta} \sigma_{22} dx &= \int_{-\delta}^{\delta} \frac{px^3 \ln t + 3pxy^2 \ln t}{(x^2 + y^2)^2} dx = p \ln t \int_{-\delta}^{\delta} \frac{x^3 + 3xy^2}{(x^2 + y^2)^2} dx \\
&= p \ln t \left[ \int_{-\delta}^{\delta} \frac{x^3}{(x^2 + y^2)^2} dx + \int_{-\delta}^{\delta} \frac{3xy^2}{(x^2 + y^2)^2} dx \right] \\
&= 0
\end{aligned}$$

Where



$$\int_{-\delta}^{\delta} \frac{x^3}{(x^2+y^2)^2} dx$$

$$u=x^2, dv=\frac{xdx}{(x^2+y^2)^2} \Rightarrow du=2xdx, v=\frac{-1}{2(x^2+y^2)}$$

$$\begin{aligned} \int_{-\delta}^{\delta} \frac{x^3}{(x^2+y^2)^2} dx &= \left. \frac{-x^2}{2(x^2+y^2)} \right|_{-\delta}^{\delta} + \int_{-\delta}^{\delta} \frac{2x}{2(x^2+y^2)} dx \\ &= \int_{-\delta}^{\delta} \frac{x}{(x^2+y^2)} dx = 0, \end{aligned}$$

$$\int_{-\delta}^{\delta} \frac{x}{(x^2+y^2)} dx$$

$$u=x^2+y^2 \Rightarrow du=2xdx$$

$$\int_{-\delta}^{\delta} \frac{x}{(x^2+y^2)} dx = \frac{1}{2} \int_{-\delta}^{\delta} \frac{du}{u} = \frac{1}{2} [\ln u]_{-\delta}^{\delta} = \frac{1}{2} \ln(x^2+y^2) \Big|_{-\delta}^{\delta} = 0$$

$$\sigma_{11} = \operatorname{Re} Z - y \operatorname{Im} Z'$$

$$\begin{aligned} &= \frac{px \ln t}{x^2+y^2} - y \frac{2pxy \ln t}{(x^2+y^2)^2} \\ &= \frac{px(x^2+y^2) \ln t - 2pxy^2 \ln t}{(x^2+y^2)^2} \\ &= \frac{px^3 \ln t + pxy^2 \ln t - 2pxy^2 \ln t}{(x^2+y^2)^2} \\ &= \frac{px^3 \ln t - pxy^2 \ln t}{(x^2+y^2)^2}. \end{aligned}$$

$$\begin{aligned} \int_{-\delta}^{\delta} \sigma_{11} dx &= \int_{-\delta}^{\delta} \frac{px^3 \ln t - pxy^2 \ln t}{(x^2+y^2)^2} dx = p \ln t \int_{-\delta}^{\delta} \frac{x^3 - xy^2}{(x^2+y^2)^2} dx \\ &= p \ln t \left[ \int_{-\delta}^{\delta} \frac{x^3}{(x^2+y^2)^2} dx - \int_{-\delta}^{\delta} \frac{xy^2}{(x^2+y^2)^2} dx \right] \\ &= 0 \end{aligned}$$

Finally, we need to verify that the stresses are consistent with a point force acting at the origin. To do this, we can evaluate the resultant force exerted by tractions acting on a circle enclosing the point force. Since the solid is in static equilibrium, the total force acting on this circular region must sum to zero. Recall that the resultant force exerted by stresses on an internal surface can be calculated as follows:

Since the traction- stress relation for plane strain/ stress is  $t_\alpha = \sigma_{\beta\alpha} \hat{n}_\beta$ , where

$\hat{n} = \frac{dX_2}{ds} \hat{e}_1 - \frac{dX_1}{ds} \hat{e}_2$  is unit outward normal and  $s$  is the arc length. Therefore,

$$t_1 = \sigma_{11} \frac{dX_2}{ds} - \sigma_{21} \frac{dX_1}{ds},$$

$$t_2 = \sigma_{12} \frac{dX_2}{ds} - \sigma_{22} \frac{dX_1}{ds},$$

We can combine as

$$t_1 - it_2 = (\sigma_{11} - i\sigma_{12}) \frac{dX_2}{ds} + (-\sigma_{12} + i\sigma_{22}) \frac{dX_1}{ds}$$

Since

$$X_1 = \frac{1}{2}(z + \bar{z}), X_2 = -\frac{1}{2}i(z - \bar{z}), \text{ then}$$

$$\begin{aligned} t_1 - it_2 &= -\frac{1}{2}i(\sigma_{11} - i\sigma_{12})\left(\frac{dz}{ds} - \frac{d\bar{z}}{ds}\right) + \frac{1}{2}(i\sigma_{22} - \sigma_{12})\left(\frac{dz}{ds} + \frac{d\bar{z}}{ds}\right) \\ &= \frac{1}{2}i(\sigma_{22} - \sigma_{11} + 2i\sigma_{12})\frac{dz}{ds} + \frac{1}{2}i(\sigma_{11} + \sigma_{22})\frac{d\bar{z}}{ds} \end{aligned}$$

By noting that from (4.4) and (4.5) ( $\sigma_{11} + \sigma_{22} = 4\text{Re}\{\phi'\}$ ,  $\sigma_{22} - \sigma_{11} + 2i\sigma_{12} = 2(\bar{z}\phi'' + \psi')$ ), we get

$$t_1 - it_2 = i(\bar{z}\phi'' + \psi') \frac{dz}{ds} + i(\phi' + \bar{\phi}') \frac{d\bar{z}}{ds}$$

By observing that

$$\frac{d}{ds}(\bar{z}\phi' + \bar{\phi} + \psi) = (\bar{z}\phi'' + \psi') \frac{dz}{ds} + (\phi' + \bar{\phi}') \frac{d\bar{z}}{ds},$$

Thus,

$$t_1 - it_2 = i \frac{d}{ds}(\bar{z}\phi' + \bar{\phi} + \psi)$$

Which is a relation for components of the traction in terms of the complex stress functions.

Now, by sitting

$$\bar{F} = i[\bar{z}\phi' + \bar{\phi} + \psi]$$

$$\bar{F} = \int (t_1 - it_2) ds$$

$$= \int \frac{d}{ds} i(\bar{z}\phi' + \bar{\phi} + \psi) ds$$

$$= [i(\bar{z}\phi' + \bar{\phi} + \psi)]$$

$$\begin{aligned}\varphi &= \frac{-i p \ln z}{2\pi}, \quad \psi = \frac{-i p \ln z}{2\pi} \\ \bar{F} &= \left[ \bar{z} \frac{p \ln z}{2\pi} + \frac{p \ln z}{2\pi} - \frac{\overline{p \ln z}}{2\pi} \right] \\ \bar{F} &= \left[ \frac{2\pi i p k}{2\pi} + \frac{p 2\pi i}{2\pi} + \frac{p 2\pi i}{2\pi} \right] = 2\pi i, k \rightarrow 0 \\ \bar{F} &= \int (t_1 - i t_2) ds = \pi i \Rightarrow t_2 = -p\end{aligned}$$

## 4.7 Concluding comments

We have presented a brief overview of the general formulation of the Kolosov-Muskhelishvili complex potentials. After this the complex potential representation of the components of stress and displacements induced by point forces was reviewed.

In the end of this chapter, we have proposed a new Westergaard's function to model the shear stresses along the crack flanks as in Figure 4.2 predicted by the CJP model which could be used for future work.

## Chapter 5 Classic Eshelby's inclusion

### 5.1 Introduction

This chapter is focused on Eshelby's technique for determining the stress, displacement and strain in regions in an infinite elastic body that undergo a change of size or shape.

### 5.2 Translation of Eshelby's solution in terms of Muskhelishvili's complex function approach to 2D elasticity

#### 5.2.1 Eshelby's inclusion: Stress and strain

Using Eshelby's analysis we have found the solution for the stress, strain and displacement fields both inside and outside the inclusion assuming the matrix is a continuum and the inclusion is a disc (circular).

Figure 5.1 shows steps of operations used by Eshelby (1957) to solve the problem of finding solution for stress and displacement fields.

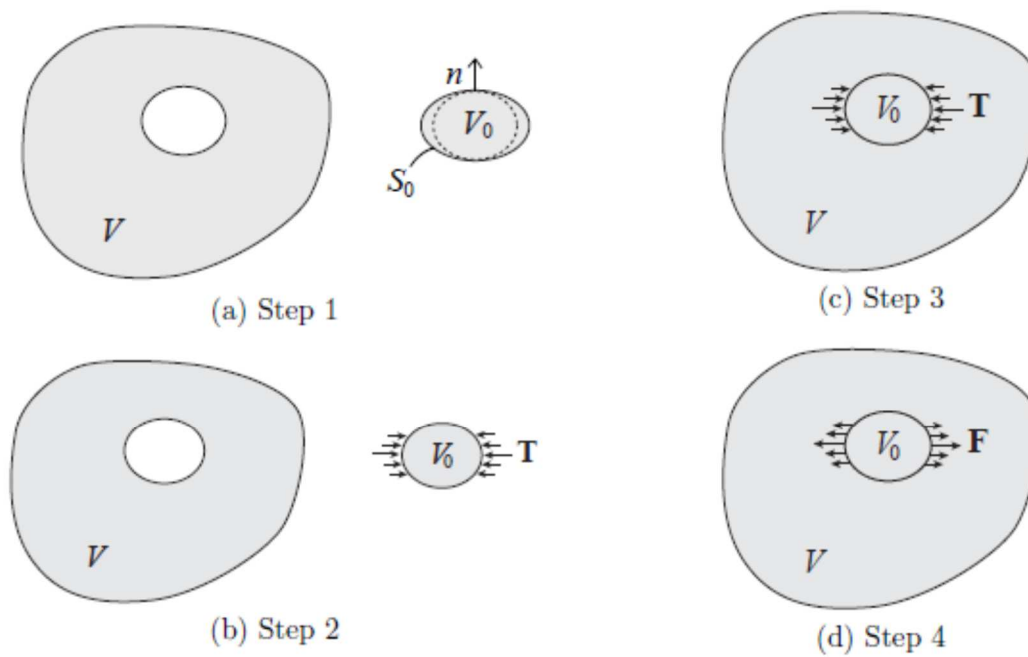


Figure 5.1: The sequence of steps of Eshelby's inclusion problem [53]

### Step 1:

Remove the inclusion from the matrix (Figure 5.1 a). The inclusion is then permanently deformed with an eigenstrain  $\varepsilon_{ij}^*$ . No external forces are applied to either the matrix or the inclusion since the inclusion is outside the matrix. The strain, stress and displacement fields in the matrix and the inclusion are given by:

matrix	Inclusion
$\varepsilon_{ij}=0$	$\varepsilon_{ij}=\varepsilon_{ij}^*$
$\sigma_{ij}=0$	$\sigma_{ij}=0$
$u_i=0$	$u_i=\varepsilon_{ij}^* x_j$

Since the strain field  $\varepsilon_{ij}$  in the inclusion is given by  $\varepsilon_{ij} = \varepsilon_{ij}^*$ , then one can give the corresponding displacement field in the inclusion in terms of Muskhelishvili's complex potential function in the following steps:

Since  $u_i = \varepsilon_{ij}^* x_j$ , then we can write  $u_1$  and  $u_2$  as

$$u_1 = \varepsilon_{1j}^* x_j = \varepsilon_{11}^* x_1 + \varepsilon_{12}^* x_2, \quad u_2 = \varepsilon_{2j}^* x_j = \varepsilon_{21}^* x_1 + \varepsilon_{22}^* x_2, \quad (5.1)$$

which can be combined as

$$u_1 - iu_2 = \varepsilon_{11}^* x_1 + \varepsilon_{12}^* x_2 - i\varepsilon_{21}^* x_1 - i\varepsilon_{22}^* x_2. \quad (5.2)$$

Noting that  $\varepsilon_{12}^*$  is symmetric, and that  $x_1 = \frac{1}{2}(z + \bar{z})$  and  $x_2 = -\frac{1}{2}i(z - \bar{z})$ , gives

$$\begin{aligned} u_1 - iu_2 &= \left[ \frac{z + \bar{z}}{2} \varepsilon_{11}^* - i(x_1 + ix_2) \varepsilon_{12}^* - i\left(\frac{z - \bar{z}}{2i}\right) \varepsilon_{22}^* \right] \\ &= \left[ \frac{z + \bar{z}}{2} \varepsilon_{11}^* - i z \varepsilon_{12}^* - \left(\frac{z - \bar{z}}{2}\right) \varepsilon_{22}^* \right]. \end{aligned} \quad (5.3)$$

We can simplify the above equation to

$$\begin{aligned}
u_1 - iu_2 &= \frac{(z+\bar{z})\varepsilon_{11}^* - (z-\bar{z})\varepsilon_{22}^* - 2iz\varepsilon_{12}^*}{2} \\
&= \frac{(\varepsilon_{11}^* - \varepsilon_{22}^* - 2i\varepsilon_{12}^*)z + (\varepsilon_{11}^* + \varepsilon_{22}^*)\bar{z}}{2} \\
&= \frac{(\varepsilon_{11}^* - \varepsilon_{22}^* - 2i\varepsilon_{12}^*)}{2}z + \frac{(\varepsilon_{11}^* + \varepsilon_{22}^*)}{2}\bar{z}. \tag{5.4}
\end{aligned}$$

Therefore,

$$\begin{aligned}
2\mu(u_1 - iu_2) &= 2\mu \left[ \frac{(\varepsilon_{11}^* - \varepsilon_{22}^* - 2i\varepsilon_{12}^*)}{2}z + \frac{(\varepsilon_{11}^* + \varepsilon_{22}^*)}{2}\bar{z} \right] \\
&= \alpha z + \beta \bar{z}, \tag{5.5}
\end{aligned}$$

where

$$\alpha = \mu(\varepsilon_{11}^* - \varepsilon_{22}^* - 2i\varepsilon_{12}^*), \beta = (\varepsilon_{11}^* + \varepsilon_{22}^*)\mu \tag{5.6}$$

Now, we calculate  $\varphi^*$  and  $\psi^*$  for the displacement.

Since  $2\mu(u_1 - iu_2) = -\bar{z}(\varphi^*)' + k\overline{\varphi^*} - \psi^*$ , from (5.5) and (5.6), we obtain

$$-\bar{z}(\varphi^*)' + k\overline{\varphi^*} - \psi^* = \alpha z + \beta \bar{z}. \tag{5.7}$$



To find  $\phi^*$  and  $\psi^*$ , we assume they are of the form

$$\phi^* = A + Cz, \quad \psi^* = B + Dz + Fz^2, \quad (5.8)$$

then

$$(\phi^*)' = C, (\psi^*)' = D + 2Fz. \quad (5.9)$$

Substituting (5.8) into (5.7) gives

$$\begin{aligned} -C\bar{z} + k(\bar{A} + \bar{C}\bar{z}) - B - Dz - Fz^2 &= \alpha z + \beta\bar{z} \\ \Rightarrow (k\bar{C} - C)\bar{z} + k\bar{A} - B - Dz - Fz^2 &= \alpha z + \beta\bar{z}. \end{aligned} \quad (5.10)$$

Equating coefficients of  $\bar{z}$  gives

$$\begin{aligned} (k\bar{C} - C) &= \beta \\ \text{Re}\{k\bar{C} - C\} &= (k-1)\text{Re}\{C\} = \beta = \mu(\varepsilon_{11}^* + \varepsilon_{22}^*) \\ \Rightarrow \text{Re}\{C\} &= \frac{\mu(\varepsilon_{11}^* + \varepsilon_{22}^*)}{(k-1)}, \end{aligned} \quad (5.11)$$

Since  $\text{Im}\{k\bar{C} - C\} = -(k+1)\text{Im}\{C\} = 0$ , then

$$C = \frac{\mu(\varepsilon_{11}^* + \varepsilon_{22}^*)}{(k-1)}. \quad (5.12)$$

By equating coefficients of  $Z$  we get

$$-D=\alpha \Rightarrow D=-\mu(\varepsilon_{11}^* -\varepsilon_{22}^* -2i\varepsilon_{12}^*), \quad (5.13)$$

and

$$K\bar{A}-B=0 \Rightarrow K\bar{A}=B, \quad (5.14)$$

$$F=0. \quad (5.15)$$

Substituting (5.12) into the equation (5.8) leads to

$$\begin{aligned} \varphi^* &= A + C z \\ &= A + \frac{\mu}{k-1}(\varepsilon_{11}^* + \varepsilon_{22}^*)z, \end{aligned} \quad (5.16)$$

also by the substituting (5.13), (5.14) and (5.15) into the equation (5.8) gives

$$\begin{aligned} \psi^* &= B + D z + F z^2 \\ &= k\bar{A} - \mu(\varepsilon_{11}^* - \varepsilon_{22}^* - 2i\varepsilon_{12}^*) z. \end{aligned} \quad (5.17)$$

## Step 2:

A traction  $T_j = \sigma_{ij} n_i = -\sigma_{ij}^* n_i$  (where  $\sigma_{ij}^* = C_{ijkl} \epsilon_{kl}^*$  and  $n_i$  is the outward normal of the inclusion) is applied to  $S_0$  (the inclusion boundary) to take the inclusion back to its original shape and size (Figure 5.1 b). Therefore, the strain, stress and displacement fields in the matrix and the inclusion are,

matrix	inclusion
$\epsilon_{ij} = 0$	$\epsilon_{ij} = \epsilon_{ij}^{el} + \epsilon_{ij}^* = 0$
$\sigma_{ij} = 0$	$\sigma_{ij} = C_{ijkl} \epsilon_{kl}^{el} = -C_{ijkl} \epsilon_{kl}^* = -\sigma_{ij}^*$
$u_i = 0$	$u_i = 0$

where  $\epsilon^{el}$  means the elastic strain of the inclusion. The stress in the inclusion can be written as

$$\sigma_{ij} = -C_{ijkl} \epsilon_{kl}^* = -\sigma_{ij}^*, \quad (5.18)$$

where

$$C_{ijkl} = \lambda^* \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}).$$

**Step 3:**

The inclusion is inserted back to the matrix (Figure 5.1 c). There is no change in the deformation fields in either the inclusion or the matrix from Step 2.

**Step 4:**

Remove the traction  $T$  on  $S_0$  (Figure 5.1 d). The change from Step 3 to Step 4 is equivalent to applying a canceling body force  $F=-T$  to the internal surface  $S_0$  of the elastic body. The strain, stress and displacement fields of the constrained field in the matrix and the inclusion are,

matrix	Inclusion
$\varepsilon_{ij} = \varepsilon_{ij}^c$	$\varepsilon_{ij} = \varepsilon_{ij}^c$
$\sigma_{ij} = \sigma_{ij}^c$	$\sigma_{ij} = \sigma_{ij}^c - \sigma_{ij}^* = C_{ijkl}(\varepsilon_{kl}^c - \varepsilon_{kl}^*)$
$u_i = u_i^c$	$u_i = u_i^c$

Let  $u_i^c(x)$  be the displacement field in response to body force  $F_j$  on  $S_0$ , then  $u_i^c(x)$  is called the constrained displacement field. Since  $F_j = -T_j = \sigma_{jk}^* n_k$ , then  $u_i^c$  can be expressed by using the Green's function for the constrained displacement of the elastic body as

$$u_i^c(x) = \int_{S_0} F_j(x) G_{ij}(x, x') ds(x') \quad (5.19)$$

From the definition of traction  $F_j = -T_j = \sigma_{jk}^* n_k$ , we can write equation (5.19) as

$$\begin{aligned} u_i^c(x) &= \int_{S_0} \sigma_{jk}^*(x) n_k(x') G_{ij}(x, x') ds(x') \\ &= \int_{S_0} [\sigma_{j1}^*(x) n_1(x') + \sigma_{j2}^*(x) n_2(x')] G_{ij}(x, x') ds(x') \\ &= \int_{S_0} [\sigma_{j1}^*(x) dx_2' - \sigma_{j2}^*(x) dx_1'] G_{ij}(x, x') \end{aligned}$$

We can separate this integral and write

$$u_i^c(x) = \int_{S_0} \sigma_{j1}^*(x) G_{ij}(x, x') dx_2' - \int_{S_0} \sigma_{j2}^*(x) G_{ij}(x, x') dx_1' \quad (5.20)$$

Therefore,

$$u_1^c(x) = \int_{S_0} [\sigma_{11}^*(x) G_{11}(x, x') + \sigma_{21}^*(x) G_{12}(x, x')] dx_2' - \int_{S_0} [\sigma_{12}^*(x) G_{11}(x, x') + \sigma_{22}^*(x) G_{12}(x, x')] dx_1', \quad (5.21)$$

and

$$u_2^c(x) = \int_{S_0} [\sigma_{11}^*(x) G_{21}(x, x') + \sigma_{21}^*(x) G_{22}(x, x')] dx_2' - \int_{S_0} [\sigma_{12}^*(x) G_{21}(x, x') + \sigma_{22}^*(x) G_{22}(x, x')] dx_1'. \quad (5.22)$$

From (2.104) the Green's function  $G_{ij}(x-x')$  is given by

$$G_{ij}(x-x') = C_1 \delta_{ij} \ln \frac{1}{|x-x'|} + C_2 \frac{(x_i - x_i')(x_j - x_j')}{|x-x'|^2},$$

where

$$C_1 = \frac{\lambda^* + 3\mu}{4\pi\mu(\lambda^* + 2\mu)}, \quad C_2 = \frac{\lambda^* + \mu}{4\pi\mu(\lambda^* + 2\mu)}.$$

We rewrite the Green's function in the complex form as follows:

$$G_{ij}(z-\zeta) = C_1 \delta_{ij} \ln \frac{1}{|z-\zeta|} + C_2 \frac{(x_i - x_i')(x_j - x_j')}{|z-\zeta|^2}.$$

Therefore,

$$G_{11}(z-\zeta) = C_1 \ln \frac{1}{|z-\zeta|} + C_2 \frac{(x_1 - x_1')^2}{|z-\zeta|^2},$$

$$G_{22}(z-\zeta) = C_1 \ln \frac{1}{|z-\zeta|} + C_2 \frac{(x_2 - x_2')^2}{|z-\zeta|^2},$$

and

$$G_{12}(z-\zeta) = G_{21}(z-\zeta) = C_2 \frac{(x_1 - x_1')(x_2 - x_2')}{|z-\zeta|^2}.$$

Now, from equations (5.21) and (5.22) we can calculate

$$\begin{aligned} u_1^c - i u_2^c &= \int_{S_0} [\sigma_{11}^* dx_2' - \sigma_{12}^* dx_1'] G_{11}(x, x') + [\sigma_{21}^* dx_2' - \sigma_{22}^* dx_1'] G_{12}(x, x') + \\ &\quad i \left( \int_{S_0} [-\sigma_{11}^* dx_2' + \sigma_{12}^* dx_1'] G_{12}(x, x') + [-\sigma_{21}^* dx_2' + \sigma_{22}^* dx_1'] G_{22}(x, x') \right) \\ &= \int_{S_0} [-\sigma_{11}^* dx_2' + \sigma_{12}^* dx_1'] [-G_{11}(x, x') + i G_{12}(x, x')] + \\ &\quad [-\sigma_{21}^* dx_2' + \sigma_{22}^* dx_1'] [-G_{12}(x, x') + i G_{22}(x, x')]. \end{aligned} \quad (5.23)$$

Putting in  $G_{11}$ ,  $G_{22}$  and  $G_{12}$  into equation (5.23), we obtain

$$\begin{aligned}
u_1^c - i u_2^c = \int_{s_0} [-\sigma_{11}^* dx_2' + \sigma_{12}^* dx_1'] & [-C_1 \ln \frac{1}{|z-\zeta|} - C_2 \frac{(x_1-x_1')^2}{|z-\zeta|^2} \\
& + i C_2 \frac{(x_1-x_1')(x_2-x_2')}{|z-\zeta|^2}] + [-\sigma_{21}^* dx_2' + \sigma_{22}^* dx_1'] [-C_2 \frac{(x_1-x_1')(x_2-x_2')}{|z-\zeta|^2} \\
& + i C_1 \ln \frac{1}{|z-\zeta|} + i C_2 \frac{(x_2-x_2')^2}{|z-\zeta|^2}].
\end{aligned}$$

Simplifying and evaluating the above expressions gives

$$\begin{aligned}
u_1^c - i u_2^c = \int_{s_0} [-\sigma_{11}^* dx_2' + \sigma_{12}^* dx_1'] & [-C_1 \ln \frac{1}{|z-\zeta|} + C_2 \frac{(x_1-x_1')}{|z-\zeta|^2} [-(x_1-x_1') + i(x_2-x_2')]] \\
& + \int_{s_0} [-\sigma_{21}^* dx_2' + \sigma_{22}^* dx_1'] [C_2 \frac{(x_2-x_2')}{|z-\zeta|^2} [-(x_1-x_1') + i(x_2-x_2')] + i C_1 \ln \frac{1}{|z-\zeta|}],
\end{aligned} \tag{5.24}$$

which simplifies to

$$\begin{aligned}
u_1^c - i u_2^c = \int_{s_0} [-\sigma_{11}^* dx_2' + \sigma_{12}^* dx_1'] & [-C_1 \ln \frac{1}{|z-\zeta|} - C_2 \frac{(x_1-x_1')}{|z-\zeta|^2} \overline{(z-\zeta)} + \\
\int_{s_0} [-\sigma_{21}^* dx_2' + \sigma_{22}^* dx_1'] & [-C_2 \frac{(x_2-x_2')}{|z-\zeta|^2} \overline{(z-\zeta)} + i C_1 \ln \frac{1}{|z-\zeta|}].
\end{aligned} \tag{5.25}$$

Equation (5.25) rearranges to give

$$\begin{aligned}
u_1^c - i u_2^c = \int_{s_0} C_1 \ln \frac{1}{|z-\zeta|} & [\sigma_{11}^* dx_2' - \sigma_{12}^* dx_1' - i \sigma_{21}^* dx_2' + i \sigma_{22}^* dx_1'] \\
& + C_2 [-\frac{(x_1-x_1')}{(z-\zeta)} (-\sigma_{11}^* dx_2' + \sigma_{12}^* dx_1') - \frac{(x_2-x_2')}{(z-\zeta)} (-\sigma_{21}^* dx_2' + \sigma_{22}^* dx_1')],
\end{aligned} \tag{5.26}$$



which reduces to

$$\begin{aligned}
 u_1^c - i u_2^c = & C_1 \int_{s_0} \ln \frac{1}{|z-\zeta|} \left[ \sigma_{11}^* \frac{d\zeta - d\bar{\zeta}}{2i} - \sigma_{12}^* \frac{d\zeta + d\bar{\zeta}}{2} - i \sigma_{21}^* \frac{d\zeta - d\bar{\zeta}}{2i} + i \sigma_{22}^* \frac{d\zeta + d\bar{\zeta}}{2} \right] \\
 & + C_2 \int_{s_0} \left[ - \frac{(x_1 - x_1')}{(z-\zeta)} \left( -\sigma_{11}^* \frac{d\zeta - d\bar{\zeta}}{2i} + \sigma_{12}^* \frac{d\zeta + d\bar{\zeta}}{2} \right) - \right. \\
 & \left. \frac{(x_2 - x_2')}{(z-\zeta)} \left( -\sigma_{21}^* \frac{d\zeta - d\bar{\zeta}}{2i} + \sigma_{22}^* \frac{d\zeta + d\bar{\zeta}}{2} \right) \right]. \quad (5.27)
 \end{aligned}$$

Simplifying equation (5.27) we get

$$\begin{aligned}
 u_1^c - i u_2^c = & C_1 \int_{s_0} \ln \frac{1}{|z-\zeta|} \left[ \frac{\sigma_{11}^* d\zeta - \sigma_{11}^* d\bar{\zeta} - \sigma_{22}^* d\zeta - \sigma_{22}^* d\bar{\zeta} - 2i \sigma_{12}^* d\zeta}{2i} \right] \\
 & + C_2 \int_{s_0} \left[ \frac{(x_1 - x_1')}{(z-\zeta)} \left( \frac{\sigma_{11}^* d\zeta - \sigma_{11}^* d\bar{\zeta} - i \sigma_{12}^* d\zeta - i \sigma_{12}^* d\bar{\zeta}}{2i} \right) + \right. \\
 & \left. \frac{(x_2 - x_2')}{(z-\zeta)} \left( \frac{\sigma_{21}^* d\zeta - \sigma_{21}^* d\bar{\zeta} - i \sigma_{22}^* d\zeta - i \sigma_{22}^* d\bar{\zeta}}{2i} \right) \right], \quad (5.28)
 \end{aligned}$$

and hence

$$\begin{aligned}
 u_1^c - i u_2^c = & C_1 \int_{s_0} \ln \frac{1}{|z-\zeta|} \left[ \frac{(-\sigma_{22}^* + \sigma_{11}^* - 2i \sigma_{12}^*) d\zeta + (-\sigma_{11}^* - \sigma_{22}^*) d\bar{\zeta}}{2i} \right] \\
 & + C_2 \int_{s_0} \left[ \frac{(z-\zeta) + \overline{(z-\zeta)}}{2(z-\zeta)} \right] \left( \frac{\sigma_{11}^* d\zeta - \sigma_{11}^* d\bar{\zeta} - i \sigma_{12}^* d\zeta - i \sigma_{12}^* d\bar{\zeta}}{2i} \right) + \\
 & \left[ \frac{(z-\zeta) - \overline{(z-\zeta)}}{2i(z-\zeta)} \right] \left( \frac{\sigma_{21}^* d\zeta - \sigma_{21}^* d\bar{\zeta} - i \sigma_{22}^* d\zeta - i \sigma_{22}^* d\bar{\zeta}}{2i} \right)
 \end{aligned}$$

$$\begin{aligned}
&= C_1 \int_{s_0} \ln \frac{1}{|z-\zeta|} \left[ \frac{(-\sigma_{22}^* + \sigma_{11}^* - 2i \sigma_{12}^*) d\zeta + (-\sigma_{11}^* - \sigma_{22}^*) d\bar{\zeta}}{2i} \right] \\
&+ C_2 \int_{s_0} \left[ \frac{\sigma_{11}^*(z-\zeta) d\zeta + \sigma_{11}^*(\bar{z}-\bar{\zeta}) d\bar{\zeta} - \sigma_{11}^*(z-\zeta) d\bar{\zeta}}{4i(z-\zeta)} + \right. \\
&\quad \left. \frac{-\sigma_{11}^*(\bar{z}-\bar{\zeta}) d\bar{\zeta} - i(z-\zeta) \sigma_{12}^* d\zeta - i(\bar{z}-\bar{\zeta}) \sigma_{12}^* d\bar{\zeta} - i(z-\zeta) \sigma_{12}^* d\bar{\zeta} - i(\bar{z}-\bar{\zeta}) \sigma_{12}^* d\zeta}{4i(z-\zeta)} + \right. \\
&\quad \left. \frac{-\sigma_{21}^*(z-\zeta) d\zeta + \sigma_{21}^*(\bar{z}-\bar{\zeta}) d\bar{\zeta} + \sigma_{21}^*(z-\zeta) d\bar{\zeta}}{4(z-\zeta)} + \frac{-\sigma_{21}^*(\bar{z}-\bar{\zeta}) d\bar{\zeta} + i \sigma_{22}^*(z-\zeta) d\zeta - i \sigma_{22}^*(\bar{z}-\bar{\zeta}) d\bar{\zeta}}{4(z-\zeta)} \right. \\
&\quad \left. + \frac{i \sigma_{22}^*(z-\zeta) d\bar{\zeta} - i \sigma_{22}^*(\bar{z}-\bar{\zeta}) d\zeta}{4(z-\zeta)} \right] \\
&= C_1 \int_{s_0} \ln \frac{1}{|z-\zeta|} \left[ \frac{(-\sigma_{22}^* + \sigma_{11}^* - 2i \sigma_{12}^*) d\zeta + (-\sigma_{11}^* - \sigma_{22}^*) d\bar{\zeta}}{2i} \right] \\
&+ C_2 \int_{s_0} \left[ \frac{[\sigma_{11}^* - \sigma_{22}^* - 2i \sigma_{12}^*](z-\zeta) d\zeta + [\sigma_{11}^* + \sigma_{22}^*](\bar{z}-\bar{\zeta}) d\bar{\zeta}}{4i(z-\zeta)} \right. \\
&\quad \left. + \frac{[-\sigma_{11}^* - \sigma_{22}^*](z-\zeta) d\bar{\zeta} + [-\sigma_{11}^* + \sigma_{22}^* - 2i \sigma_{12}^*](\bar{z}-\bar{\zeta}) d\zeta}{4i(z-\zeta)} \right] \\
&= C_1 \left[ \frac{1}{2i} (-\Sigma^*) \int_{s_0} \ln \frac{1}{|z-\zeta|} d\zeta - \frac{1}{2i} T^* \int_{s_0} \ln \frac{1}{|z-\zeta|} d\bar{\zeta} \right] \\
&+ C_2 \left[ \frac{1}{4i} (-\Sigma^*) \int_{s_0} d\zeta + \frac{1}{4i} T^* \int_{s_0} \frac{(\bar{z}-\bar{\zeta})}{(z-\zeta)} d\zeta - \right. \\
&\quad \left. \frac{1}{4i} T^* \int_{s_0} d\bar{\zeta} + \frac{1}{4i} (\bar{\Sigma}^*) \int_{s_0} \frac{(z-\zeta)}{(\bar{z}-\bar{\zeta})} d\bar{\zeta} \right]. \tag{5.29}
\end{aligned}$$

In following, we will need to calculate the integrals:

$$\int_{s_0} \ln \frac{1}{|z-\zeta|} d\zeta, \int_{s_0} \ln \frac{1}{|z-\zeta|} d\bar{\zeta}, \int_{s_0} \frac{(\bar{z}-\bar{\zeta})}{(z-\zeta)} d\zeta \text{ and } \int_{s_0} \frac{(z-\zeta)}{(\bar{z}-\bar{\zeta})} d\bar{\zeta}.$$

Using the property of the logarithm, one can find the integral  $\int_{S_0} \ln \frac{1}{|z-\zeta|} d\zeta$  as

$$\begin{aligned}
 \int_{S_0} \ln \frac{1}{|z-\zeta|} d\zeta &= \int_{S_0} [\ln 1 - \ln |z-\zeta|] d\zeta = - \int_{S_0} \ln |z-\zeta| d\zeta \\
 &= - \int \ln [(z-\zeta)(\overline{z-\zeta})]^{\frac{1}{2}} d\zeta = - \frac{1}{2} \int \ln [(z-\zeta)(\overline{z-\zeta})] d\zeta \\
 &= - \frac{1}{2} \int [\ln(z-\zeta) + \ln(\overline{z-\zeta})] d\zeta \quad (5.30) \\
 &= - \frac{1}{2} \int \ln(z-\zeta) d\zeta - \frac{1}{2} \int \ln(\overline{z-\zeta}) d\zeta \\
 &= - \frac{1}{2} \int \ln(z-\zeta) d\zeta - \frac{1}{2} \int \overline{\ln(z-\zeta)} d\zeta.
 \end{aligned}$$

If  $|\zeta| < |z|$ , then

$$\ln(z-\zeta) = \ln[z(1-\frac{\zeta}{z})] = \ln z + \ln(1-\frac{\zeta}{z}).$$

Now the integral can be calculated using the above expression as

$$\int \ln(z-\zeta) d\zeta = \int \ln[z(1-\frac{\zeta}{z})] d\zeta = \int \ln z d\zeta + \int \ln(1-\frac{\zeta}{z}) d\zeta. \quad (5.31)$$

The Taylor series of  $\ln(1-\frac{\zeta}{z})$  is

$$\ln\left(1-\frac{\zeta}{z}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n(-\zeta)^{n+1}}{(n+1)z^{n+1}}, \quad |\zeta|<|z|,$$

which can be substituted into this integral to give

$$\int \ln\left(1-\frac{\zeta}{z}\right) d\zeta = \int \sum_{n=0}^{\infty} \frac{(-1)^n (-\zeta)^{n+1}}{(n+1)z^{n+1}} d\zeta = \sum_{n=0}^{\infty} \frac{(-1)^{2n+1}}{(n+1)z^{n+1}} \int (\zeta)^{n+1} d\zeta.$$

We work on the circle  $|\zeta|=r$ , so let  $\zeta=re^{i\theta}$ ,  $d\zeta=rie^{i\theta}d\theta$  then  $\zeta^{n+1}=r^{n+1}e^{i(n+1)\theta}$ . The complex integral is

$$\begin{aligned} \int \zeta^{n+1} d\zeta &= \int_0^{2\pi} r^{(n+1)} e^{i(n+1)\theta} rie^{i\theta} d\theta = r^{(n+2)} i \int_0^{2\pi} e^{i(n+2)\theta} d\theta \\ &= r^{(n+2)} i \left[ \frac{e^{i(n+2)\theta}}{i(n+2)} \right]_0^{2\pi} = \frac{r^{(n+2)}}{(n+2)} [e^{2i(n+2)\pi} - e^0] = 0, \end{aligned}$$

where

$$e^{2i(n+2)\pi} = \cos(2(n+2)\pi) + i\sin(2(n+2)\pi) = 1 + i(0) = 1.$$

Therefore,

$$\int \ln\left(1 - \frac{\zeta}{z}\right) d\zeta = \int \sum_{n=0}^{\infty} \frac{(-1)^n (-\zeta)^{n+1}}{(n+1)z^{n+1}} d\zeta = \sum_{n=0}^{\infty} \frac{(-1)^{2n+1}}{(n+1)z^{n+1}} \int (\zeta)^{n+1} d\zeta = 0.$$

By using the preceding result, equation (5.31) becomes

$$\int \ln(z - \zeta) d\zeta = \int \ln\left[z\left(1 - \frac{\zeta}{z}\right)\right] d\zeta = \int \ln z d\zeta + \int \ln\left(1 - \frac{\zeta}{z}\right) d\zeta = 0, \quad (5.32)$$

in which

$$\begin{aligned} \int \ln(z) d\zeta &= \ln(z) \int d\zeta = \ln(z) \int_0^{2\pi} r i e^{i\theta} d\theta = r i \ln(z) \int_0^{2\pi} e^{i\theta} d\theta \\ &= r i \ln(z) \left[ \frac{e^{i\theta}}{i} \right]_0^{2\pi} = r \ln(z) [e^{2\pi i} - e^0] = 0. \end{aligned}$$

In a similar way we can represent the integral as

$$\int \ln(z - \zeta) d\bar{\zeta} = \int \ln\left[z\left(1 - \frac{\zeta}{z}\right)\right] d\bar{\zeta} = \int \ln z d\bar{\zeta} + \int \ln\left(1 - \frac{\zeta}{z}\right) d\bar{\zeta}. \quad (5.33)$$

By using Taylor series of  $\ln\left(1 - \frac{\zeta}{z}\right)$  we can evaluate this integral as

$$\begin{aligned}
\int \ln\left(1 - \frac{\zeta}{z}\right) d\bar{\zeta} &= \int \sum_{n=0}^{\infty} \frac{(-1)^n (-\zeta)^{n+1}}{(n+1)z^{n+1}} d\bar{\zeta} = \sum_{n=0}^{\infty} \frac{(-1)^{2n+1}}{(n+1)z^{n+1}} \int (\zeta)^{n+1} d\bar{\zeta} \\
&= \frac{(-1)}{z} \int \zeta d\bar{\zeta} + \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{(n+1)z^{n+1}} \int (\zeta)^{n+1} d\bar{\zeta}, \quad |\zeta| < |z|.
\end{aligned} \tag{5.34}$$

As before we can work on the circle  $|\zeta|=r$ , so taking  $\zeta=re^{i\theta}$  and  $\zeta^{n+1}=r^{n+1}e^{i(n+1)\theta}$ , then

$d\zeta=re^{i\theta}d\theta \Rightarrow d\bar{\zeta}=-re^{-i\theta}d\theta$  and substituting these into equation (5.34) gives

$$\int \ln\left(1 - \frac{\zeta}{z}\right) d\bar{\zeta} = \frac{2\pi ir^2}{z} = \frac{2\pi i |\zeta|^2}{z},$$

in which

$$\frac{(-1)}{z} \int \zeta d\bar{\zeta} = \frac{(-1)}{z} \int_0^{2\pi} re^{i\theta} (-re^{-i\theta} d\theta) = \frac{i}{z} \int_0^{2\pi} r^2 d\theta = \frac{2\pi ir^2}{z},$$

and

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{(n+1)z^{n+1}} \int \zeta^{n+1} d\bar{\zeta} &= \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{(n+1)z^{n+1}} \int_0^{2\pi} r^{n+1} e^{i(n+1)\theta} (-re^{-i\theta}) d\theta \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{(n+1)z^{n+1}} (-r^{n+2}) i \int_0^{2\pi} e^{in\theta} d\theta \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{(n+1)z^{n+1}} (-r^{n+2}) i \left[ \frac{e^{in\theta}}{in} \right]_0^{2\pi} = \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{(n+1)z^{n+1}} \left( \frac{-r^{n+2}}{n} [e^{2in\pi} - e^0] \right) = 0.
\end{aligned}$$

Since

$$\begin{aligned}
\int \ln(z) d\bar{\zeta} &= \ln(z) \int d\bar{\zeta} = \ln(z) \int_0^{2\pi} (-rie^{-i\theta} d\theta) = -r i \ln(z) \int_0^{2\pi} e^{-i\theta} d\theta \\
&= -r i \ln(z) \left[ \frac{e^{-i\theta}}{-i} \right]_0^{2\pi} = r \ln(z) [e^{2\pi i} - e^0] = 0,
\end{aligned}$$

then it follows from equation (5.33) and (5.34) that

$$\begin{aligned}
\int \ln(z-\zeta) d\bar{\zeta} &= \int \ln\left[z\left(1-\frac{\zeta}{z}\right)\right] d\bar{\zeta} = \int \ln z d\bar{\zeta} + \int \ln\left(1-\frac{\zeta}{z}\right) d\bar{\zeta} \\
&= \frac{2\pi i r^2}{z}.
\end{aligned} \tag{5.35}$$

By substituting from equations (5.32) and (5.35), the equation (5.30) reduces to

$$\begin{aligned}
\int_{s_0} \ln \frac{1}{|z-\zeta|} d\zeta &= -\frac{1}{2} \left[ \int \ln(z-\zeta) d\zeta + \overline{\int \ln(z-\zeta) d\bar{\zeta}} \right] \\
&= -\frac{1}{2} \left( \frac{-2\pi i r^2}{\bar{z}} \right) = \left( \frac{\pi i r^2}{\bar{z}} \right), \text{ for } r < |z|.
\end{aligned} \tag{5.36}$$

In a similar way the equation (5.30) can be calculated when  $|z| < r$  as

$$\int \ln(z-\zeta) d\zeta = \int \ln(-\zeta) d\zeta + \int \ln\left(1-\frac{z}{\zeta}\right) d\zeta = 2i\pi(r-z), \tag{5.37}$$

in more detail,

$$\ln(z-\zeta)=\ln[\zeta(\frac{z}{\zeta}-1)]=\ln[(-\zeta)(1-\frac{z}{\zeta})]=\ln(-\zeta)+\ln(1-\frac{z}{\zeta}), |z|<r, \quad (5.38)$$

and

$$\begin{aligned} \int \ln(-\zeta) d\zeta &= \int_0^{2\pi} \ln(-re^{i\theta})(ire^{i\theta} d\theta) = \int_0^{2\pi} [\ln(-r) + \ln e^{i\theta}](ire^{i\theta} d\theta) \\ &= \int_0^{2\pi} \ln(-r)(ire^{i\theta} d\theta) + \int_0^{2\pi} \ln e^{i\theta}(ire^{i\theta} d\theta) \\ &= ir \ln(-r) \int_0^{2\pi} e^{i\theta} d\theta + ir \int_0^{2\pi} e^{i\theta} (i\theta \ln e) d\theta \\ &= ir \ln(-r) \left[ \frac{e^{i\theta}}{i} \right]_0^{2\pi} - r \int_0^{2\pi} \theta e^{i\theta} d\theta \\ &= -r \int_0^{2\pi} \theta e^{i\theta} d\theta = -r(-2i\pi). \end{aligned} \quad (5.39)$$

Since Taylor series of  $\ln(1-\frac{z}{\zeta})$  is

$$\ln(1-\frac{z}{\zeta}) = \sum_{n=0}^{\infty} \frac{(-1)^n (-z)^{n+1}}{(n+1)\zeta^{n+1}}, \quad |z| < r,$$

then



$$\begin{aligned}
\int \ln\left(1 - \frac{z}{\zeta}\right) d\zeta &= \int \sum_{n=0}^{\infty} \frac{(-1)^n (-z)^{n+1}}{(n+1)\zeta^{n+1}} d\zeta \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (-z)^{n+1}}{(n+1)} \int (\zeta)^{-(n+1)} d\zeta \\
&= (-z) \int (\zeta)^{-1} d\zeta + \sum_{n=1}^{\infty} \frac{(-1)^n (-z)^{n+1}}{(n+1)} \int (\zeta)^{-(n+1)} d\zeta \\
&= (-z) \int (\zeta)^{-1} d\zeta + \sum_{n=1}^{\infty} \frac{(-1)^n (-z)^{n+1}}{(n+1)} \int (\zeta)^{-(n+1)} d\zeta \\
&= (-z) \int_0^{2\pi} r^{-1} e^{-i\theta} (rie^{i\theta}) d\theta + \sum_{n=1}^{\infty} \frac{(-1)^n (-z)^{n+1}}{(n+1)} \int_0^{2\pi} r^{-(n+1)} e^{-i(n+1)\theta} (rie^{i\theta}) d\theta \\
&= (-z)i \int_0^{2\pi} d\theta + \sum_{n=1}^{\infty} \frac{(-1)^n (-z)^{n+1}}{(n+1)} r^{-n} i \int_0^{2\pi} e^{-ni\theta} d\theta \\
&= -2i\pi z + (r^{-n} i) \left[ \frac{e^{-ni\theta}}{-ni} \right]_0^{2\pi} \\
&= -2i\pi z + \frac{r^{-n} i}{-ni} [e^{-2ni\pi} - e^0] = -2i\pi z,
\end{aligned} \tag{5.40}$$

where

$$e^{-2ni\pi} = \cos(-2n\pi) + i\sin(-2n\pi) = 1 + i(0) = 1.$$

Since

$$\int \ln(\overline{z-\zeta}) d\zeta = \overline{\int \ln(z-\zeta) d\overline{\zeta}}, \tag{5.41}$$

we can find  $\int \ln(z-\zeta) d\overline{\zeta}$  as follows:

$$\int \ln(z-\zeta) d\bar{\zeta} = \int \ln(-\zeta) d\bar{\zeta} + \int \ln(1-\frac{z}{\zeta}) d\bar{\zeta}, \quad (5.42)$$

where

$$\begin{aligned} \int \ln(-\zeta) d\bar{\zeta} &= \int_0^{2\pi} \ln(-re^{i\theta}) (-ire^{-i\theta} d\theta) = \int_0^{2\pi} [\ln(-r) + \ln e^{i\theta}] (-ire^{-i\theta} d\theta) \\ &= \int_0^{2\pi} \ln(-r) (-ire^{-i\theta} d\theta) + \int_0^{2\pi} \ln e^{i\theta} (-ire^{-i\theta} d\theta) \\ &= -ir \ln(-r) \int_0^{2\pi} e^{-i\theta} d\theta - ir \int_0^{2\pi} e^{-i\theta} (i\theta \ln e) d\theta \\ &= -ir \ln(-r) \left[ \frac{e^{-i\theta}}{-i} \right]_0^{2\pi} + r \int_0^{2\pi} \theta e^{-i\theta} d\theta = r \int_0^{2\pi} \theta e^{-i\theta} d\theta = r(2i\pi). \end{aligned} \quad (5.43)$$

The Taylor series of  $\ln(1-\frac{z}{\zeta})$  gives

$$\begin{aligned} \int \ln(1-\frac{z}{\zeta}) d\bar{\zeta} &= \int \sum_{n=0}^{\infty} \frac{(-1)^n (-z)^{n+1}}{(n+1)\zeta^{n+1}} d\bar{\zeta} \\ &= (-z) \int \zeta^{-1} d\bar{\zeta} + \sum_{n=1}^{\infty} \frac{(-1)^n (-z)^{n+1}}{(n+1)} \int (\zeta)^{-(n+1)} d\bar{\zeta} \\ &= (-z) \int r^{-1} e^{-i\theta} (-ire^{-i\theta}) d\theta + \sum_{n=1}^{\infty} \frac{(-1)^n (-z)^{n+1}}{(n+1)} \int_0^{2\pi} r^{-(n+1)} e^{-i(n+1)\theta} (-ire^{-i\theta}) d\theta \\ &= zi \int_0^{2\pi} e^{-2i\theta} d\theta + \sum_{n=1}^{\infty} \frac{(-1)^n (-z)^{n+1}}{(n+1)} (-r^{-n} i) \int_0^{2\pi} e^{-(n+2)i\theta} d\theta \\ &= zi \left[ \frac{e^{-2i\theta}}{-2i} \right]_0^{2\pi} + \sum_{n=1}^{\infty} \frac{(-1)^n (-z)^{n+1}}{(n+1)} (-r^{-n} i) \left[ \frac{e^{-(n+2)i\theta}}{-(n+2)i} \right]_0^{2\pi} = 0. \end{aligned} \quad (5.44)$$

By substituting (5.43) and (5.44) into (5.42) we get

$$\begin{aligned} \int \ln(z-\zeta) d\bar{\zeta} &= \int \ln(-\zeta) d\bar{\zeta} + \int \ln\left(1-\frac{z}{\zeta}\right) d\bar{\zeta} \\ &= 2i\pi|\zeta| \end{aligned} \quad (5.45)$$

Now returning to the equation (5.41), we obtain

$$\int \ln(\overline{z-\zeta}) d\zeta = \overline{\int \ln(z-\zeta) d\bar{\zeta}} = \overline{2i\pi r} = -2i\pi r. \quad (5.46)$$

From (5.30) we have

$$\begin{aligned} \int_{s_0} \ln \frac{1}{|z-\zeta|} d\zeta &= -\frac{1}{2} \int \ln(z-\zeta) d\zeta - \frac{1}{2} \int \ln(\overline{z-\zeta}) d\zeta \\ &= -\frac{1}{2} [2i\pi(|\zeta|-z) + (-2i\pi|\zeta|)] = i\pi z, \quad \text{for } |z| < r. \end{aligned} \quad (5.47)$$

We deduce that

$$\int_{s_0} \ln \frac{1}{|z-\zeta|} d\zeta = \begin{cases} i\pi z, & \text{for } |z| < r \\ \frac{i\pi r^2}{z}, & \text{for } |z| > r \end{cases}. \quad (5.48)$$

We now calculate the integral  $\int_{S_0} \frac{\overline{(z-\zeta)}}{(z-\zeta)} d\zeta$  using Taylor series by

$$\begin{aligned} \int_{S_0} \frac{\overline{(z-\zeta)}}{(z-\zeta)} d\zeta &= \int_{S_0} \overline{(z-\zeta)} \frac{1}{(z-\zeta)} d\zeta = \int \overline{(z-\zeta)} \frac{1}{z(1-\frac{\zeta}{z})} d\zeta, \quad |z| > r \\ &= \int \overline{(z-\zeta)} \sum_{n=0}^{\infty} \frac{1}{z} \left(\frac{\zeta}{z}\right)^n d\zeta = \int (\overline{z}-\overline{\zeta}) \sum_{n=0}^{\infty} \frac{\zeta^n}{z^{n+1}} d\zeta. \end{aligned} \quad (5.49)$$

Consider the circle in the complex plane:  $\zeta = re^{i\theta}$ ,  $d\zeta = rie^{i\theta} d\theta$  then  $\overline{z-\zeta} = \overline{z} - re^{-i\theta}$ . The complex integral is

$$\begin{aligned} \int_{S_0} \frac{\overline{(z-\zeta)}}{(z-\zeta)} d\zeta &= \int (\overline{z}-\overline{\zeta}) \sum_{n=0}^{\infty} \frac{\zeta^n}{z^{n+1}} d\zeta = \int (\overline{z}-\overline{\zeta}) \frac{1}{z} d\zeta + \int (\overline{z}-\overline{\zeta}) \sum_{n=1}^{\infty} \frac{\zeta^n}{z^{n+1}} d\zeta \\ &= \frac{1}{z} \int_0^{2\pi} (\overline{z} - re^{-i\theta}) (rie^{i\theta} d\theta) + \sum_{n=1}^{\infty} \int_0^{2\pi} (\overline{z} - re^{-i\theta}) \frac{(re^{i\theta})^n}{z^{n+1}} (rie^{i\theta} d\theta) \quad (5.50) \\ &= \frac{1}{z} \int_0^{2\pi} [\overline{z} rie^{i\theta} - ir^2] d\theta + \sum_{n=1}^{\infty} \int_0^{2\pi} \frac{\overline{z} i r^{(n+1)} e^{i(n+1)\theta}}{z^{n+1}} d\theta - \int_0^{2\pi} \frac{r^{n+2} i e^{in\theta}}{z^{n+1}} d\theta \\ &= \frac{\overline{z} ri}{z} \int_0^{2\pi} e^{i\theta} d\theta - \frac{ir^2}{z} \int_0^{2\pi} d\theta + \sum_{n=1}^{\infty} \frac{\overline{z} i r^{(n+1)}}{z^{n+1}} \int_0^{2\pi} e^{i(n+1)\theta} d\theta - \frac{ir^{n+2}}{z^{n+1}} \int_0^{2\pi} e^{in\theta} d\theta \\ &= \frac{\overline{z} ri}{z} \left[ \frac{e^{i\theta}}{i} \right]_0^{2\pi} - \frac{ir^2}{z} [\theta]_0^{2\pi} + \sum_{n=1}^{\infty} \frac{\overline{z} i r^{(n+1)}}{z^{n+1}} \left( \frac{e^{i(n+1)\theta}}{i(n+1)} \right) \Big|_0^{2\pi} - \frac{ir^{n+2}}{z^{n+1}} \left( \frac{e^{in\theta}}{in} \right) \Big|_0^{2\pi} \\ &= -\frac{ir^2}{z} (2\pi) + \sum_{n=1}^{\infty} \left( \frac{\overline{z} i r^{(n+1)}}{z^{n+1}} \left[ \frac{e^{i(n+1)2\pi}}{i(n+1)} - \frac{e^0}{i(n+1)} \right] - \frac{ir^{n+2}}{z^{n+1}} \left[ \frac{e^{i2n\pi}}{in} - \frac{e^0}{in} \right] \right) \\ &= -\frac{2i\pi r^2}{z}. \end{aligned}$$

Now we want to calculate the integral  $\int_{s_0} \frac{\overline{(z-\zeta)}}{(z-\zeta)} d\zeta$  when  $|z| < r$ . To do this, we can use similar steps to the above as

$$\begin{aligned} \int_{s_0} \frac{\overline{(z-\zeta)}}{(z-\zeta)} d\zeta &= \int_{s_0} \overline{(z-\zeta)} \frac{1}{(z-\zeta)} d\zeta = \int_{s_0} \overline{(z-\zeta)} \frac{-1}{\zeta(1-\frac{z}{\zeta})} d\zeta, \quad |z| < r \\ &= \int_{s_0} \overline{(z-\zeta)} \sum_{n=0}^{\infty} \frac{-1}{\zeta} \left(\frac{z}{\zeta}\right)^n d\zeta = \int_{s_0} \overline{(z-\zeta)} \sum_{n=0}^{\infty} \frac{(-1)z^n}{\zeta^{n+1}} d\zeta. \end{aligned} \quad (5.51)$$

Let  $\zeta = re^{i\theta} \Rightarrow d\zeta = rie^{i\theta} d\theta$ ,  $\overline{z-\zeta} = \overline{z} - \overline{\zeta} = \overline{z} - re^{-i\theta}$ . The complex integral then can be written as

$$\begin{aligned} \int_{s_0} \frac{\overline{(z-\zeta)}}{(z-\zeta)} d\zeta &= \int_{s_0} (\overline{z} - \overline{\zeta}) \frac{(-1)}{\zeta} d\zeta + \int_{s_0} (\overline{z} - \overline{\zeta}) \sum_{n=1}^{\infty} \frac{(-1)z^n}{\zeta^{n+1}} d\zeta \\ &= (-1) \int_{s_0} (\overline{z}\zeta^{-1} - \overline{\zeta}\zeta^{-1}) d\zeta + (-1) \sum_{n=1}^{\infty} \int_{s_0} (\overline{z} - \overline{\zeta}) \frac{z^n}{\zeta^{n+1}} d\zeta \\ &= - \int_0^{2\pi} \overline{z} r^{-1} e^{-i\theta} (rie^{i\theta} d\theta) + \int_0^{2\pi} re^{-i\theta} r^{-1} e^{-i\theta} (rie^{i\theta} d\theta) + (-1) \sum_{n=0}^{\infty} \int_0^{2\pi} (\overline{z} - re^{-i\theta}) \frac{z^n}{(re^{i\theta})^{n+1}} (rie^{i\theta} d\theta) \\ &= - \int_0^{2\pi} \overline{z} r^{-1} e^{-i\theta} (rie^{i\theta} d\theta) + \int_0^{2\pi} re^{-i\theta} r^{-1} e^{-i\theta} (rie^{i\theta} d\theta) + \\ &\quad (-1) \sum_{n=1}^{\infty} z^n \left[ \int_0^{2\pi} \overline{z} i r^{-n} e^{-in\theta} d\theta - \int_0^{2\pi} r^{-n+1} i e^{-i(n+1)\theta} d\theta \right] \quad (5.52) \\ &= -i\overline{z} \int_0^{2\pi} d\theta + ri \int_0^{2\pi} e^{-i\theta} d\theta + \sum_{n=1}^{\infty} [(-1)\overline{z} z^n i r^{-n} \int_0^{2\pi} e^{-in\theta} d\theta + i r^{-n+1} z^n \int_0^{2\pi} e^{-i(n+1)\theta} d\theta] \\ &= -2i\pi\overline{z} + ri \left( \frac{e^{-i\theta}}{-i} \right) \Big|_0^{2\pi} + \sum_{n=0}^{\infty} [(-1)\overline{z} z^n i r^{-n} \left( \frac{e^{-in\theta}}{-in} \right) \Big|_0^{2\pi} + i r^{-n+1} z^n \left( \frac{e^{-i(n+1)\theta}}{-i(n+1)} \right) \Big|_0^{2\pi}] \\ &= -2i\pi\overline{z}. \end{aligned}$$

From the above we conclude the following:

$$\int_{s_0} \frac{\overline{(z-\zeta)}}{(z-\zeta)} d\zeta = \begin{cases} -2i\pi\bar{z}, & |z| < r \\ \frac{-2i\pi r^2}{z}, & |z| > r \end{cases} \quad (5.53)$$

Now, we calculate  $\int_{s_0} \frac{\overline{(z-\zeta)}}{(z-\zeta)} d\bar{\zeta}$ . Let  $\bar{\zeta} = \frac{r^2}{\zeta} \Rightarrow d\bar{\zeta} = \frac{-r^2}{\zeta^2} d\zeta$ . Then

$$\begin{aligned} \int_{s_0} \frac{\overline{(z-\zeta)}}{(z-\zeta)} d\bar{\zeta} &= \int \frac{\bar{z} - \bar{\zeta}}{z - \zeta} d\bar{\zeta} = \int \frac{\bar{z} - (\frac{r^2}{\zeta})}{z - \zeta} (\frac{-r^2}{\zeta^2} d\zeta) \\ &= \int \frac{\zeta\bar{z} - r^2}{\zeta(z-\zeta)} (\frac{-r^2}{\zeta^2} d\zeta) = \int \frac{-\zeta\bar{z}r^2 + r^4}{\zeta^3(z-\zeta)} d\zeta \\ &= \int \frac{-r^2\zeta\bar{z}}{\zeta^3(z-\zeta)} d\zeta + \int \frac{r^4}{\zeta^3(z-\zeta)} d\zeta \\ &= r^2\bar{z} \int \frac{d\zeta}{\zeta^2(\zeta-z)} - r^4 \int \frac{d\zeta}{\zeta^3(\zeta-z)}. \end{aligned} \quad (5.54)$$

For  $|z| < r$

$$\int \frac{d\zeta}{\zeta^2(\zeta-z)} = \int \frac{A}{\zeta^2} d\zeta + \int \frac{B}{\zeta-z} d\zeta, \quad (5.55)$$

where

$$\begin{aligned}
A &= \lim_{\zeta \rightarrow 0} \left[ \frac{1}{(2-1)!} \frac{d^{2-1}}{d\zeta^{2-1}} [(\zeta-0)^2 \frac{1}{\zeta^2 (\zeta-z)}] \right] \\
&= \lim_{\zeta \rightarrow 0} \left[ \frac{d}{d\zeta} \left[ (\zeta^2) \frac{1}{\zeta^2 (\zeta-z)} \right] \right] = \lim_{\zeta \rightarrow 0} \left( \frac{-1}{(\zeta-z)^2} \right) \\
&= \frac{-1}{z^2},
\end{aligned}$$

and

$$\begin{aligned}
B &= \lim_{\zeta \rightarrow z} \left[ (\zeta-z) \frac{1}{\zeta^2 (\zeta-z)} \right] = \frac{1}{z^2} \\
\int \frac{d\zeta}{\zeta^2 (\zeta-z)} &= \int \frac{A}{\zeta^2} d\zeta + \int \frac{B}{\zeta-z} d\zeta \\
&= \int \frac{-1/z^2}{\zeta^2} d\zeta + \int \frac{1/z^2}{\zeta-z} d\zeta \\
&= 2i\pi(-1/z^2) + 2i\pi(1/z^2) = 0.
\end{aligned}$$

Now, we calculate  $\int \frac{d\zeta}{\zeta^3 (\zeta-z)}$  as

$$\int \frac{d\zeta}{\zeta^3 (\zeta-z)} = \int \frac{A}{\zeta^3} d\zeta + \int \frac{B}{\zeta-z} d\zeta,$$

where

$$\begin{aligned}
A &= \lim_{\zeta \rightarrow 0} \left[ \frac{1}{(3-1)!} \frac{d^{3-1}}{d\zeta^{3-1}} [(\zeta-0)^3 \frac{1}{\zeta^3 (\zeta-z)}] \right] \\
&= \lim_{\zeta \rightarrow 0} \left[ \frac{1}{2!} \frac{d^2}{d\zeta^2} \left( \frac{1}{(\zeta-z)} \right) \right] = \lim_{\zeta \rightarrow 0} \left[ \frac{1}{2!} \frac{d}{d\zeta} \left( \frac{-1}{(\zeta-z)^2} \right) \right] \\
&= \lim_{\zeta \rightarrow 0} \left[ \frac{1}{2!} \frac{d}{d\zeta} \left( \frac{-1}{(\zeta-z)^2} \right) \right] = \lim_{\zeta \rightarrow 0} \left[ \frac{1}{2!} \left( \frac{2}{(\zeta-z)^3} \right) \right] \\
&= \lim_{\zeta \rightarrow 0} \frac{1}{(\zeta-z)^3} = \frac{-1}{z^3},
\end{aligned}$$

and

$$\begin{aligned}
B &= \lim_{\zeta \rightarrow z} \left[ (\zeta-z) \frac{1}{\zeta^3 (\zeta-z)} \right] = \frac{1}{z^3} \\
\int \frac{d\zeta}{\zeta^3 (\zeta-z)} &= \int \frac{1/z^3}{\zeta^3} d\zeta + \int \frac{1/z^3}{\zeta-z} d\zeta = 2i\pi \left( \frac{-1}{z^3} \right) + 2i\pi \left( \frac{1}{z^3} \right) = 0 \\
\int_{s_0} \frac{\overline{(z-\zeta)}}{(z-\zeta)} d\bar{\zeta} &= r^2 \bar{z} \int \frac{d\zeta}{\zeta^2 (\zeta-z)} - r^4 \int \frac{d\zeta}{\zeta^3 (\zeta-z)} \\
&= r^2 \bar{z}(0) - r^4(0) = 0, r > |z|.
\end{aligned}$$

If  $|z| > r$ , then

$$\int_{s_0} \frac{\overline{(z-\zeta)}}{(z-\zeta)} d\bar{\zeta} = r^2 \bar{z} \int \frac{d\zeta}{\zeta^2 (\zeta-z)} - r^4 \int \frac{d\zeta}{\zeta^3 (\zeta-z)}.$$

Since



$$\int \frac{d\zeta}{\zeta^2(\zeta-z)} = \int \frac{A}{\zeta^2} d\zeta + \int \frac{B}{\zeta-z} d\zeta = 2i\pi(\cdot/\zeta^2) + 0 = 2i\pi(\cdot/\zeta^2),$$

and

$$\int \frac{d\zeta}{\zeta^3(\zeta-z)} = \int \frac{A}{\zeta^3} d\zeta + \int \frac{B}{\zeta-z} d\zeta = 2i\pi(\frac{-1}{z^3}) + 0 = 2i\pi(\frac{-1}{z^3}),$$

then

$$\begin{aligned} \int_{s_0} \frac{\overline{(z-\zeta)}}{(z-\zeta)} d\bar{\zeta} &= r^2 \bar{z} \int \frac{d\zeta}{\zeta^2(\zeta-z)} - r^4 \int \frac{d\zeta}{\zeta^3(\zeta-z)} \\ &= r^2 \bar{z} [2i\pi(\cdot/\zeta^2)] - r^4 [2i\pi(\cdot/\zeta^3)] \\ &= \frac{-2i\pi r^2}{z^2} (\bar{z} - \frac{r^2}{z}), \quad |z| > r. \end{aligned} \quad (5.56)$$

We can conclude the above as the following:

$$\int_{s_0} \frac{\overline{(z-\zeta)}}{(z-\zeta)} d\bar{\zeta} = \begin{cases} 0, & |z| < r \\ \frac{-2i\pi r^2}{z^2} (\bar{z} - \frac{r^2}{z}), & |z| > r \end{cases}. \quad (5.57)$$

Now

$$\begin{aligned}
u_1^c - iu_2^c = & \frac{\lambda^* + 3\mu}{4\pi\mu(\lambda^* + 2\mu)} \left[ \frac{1}{2i} \Sigma^* \int_{s_0} \ln \frac{1}{|z-\zeta|} d\zeta + \frac{1}{2i} T^* \int_{s_0} \ln \frac{1}{|z-\bar{\zeta}|} d\bar{\zeta} \right] \\
& + \frac{\lambda^* + \mu}{4\pi\mu(\lambda^* + 2\mu)} \left[ \frac{1}{4i} \Sigma^* \int_{s_0} d\zeta - \frac{1}{4i} T^* \int_{s_0} \frac{\overline{(z-\zeta)}}{(z-\zeta)} d\zeta + \right. \\
& \left. \frac{1}{4i} T^* \int_{s_0} d\bar{\zeta} + \frac{1}{4i} (-\overline{\Sigma^*}) \int_{s_0} \frac{(z-\bar{\zeta})}{(z-\bar{\zeta})} d\bar{\zeta} \right].
\end{aligned} \tag{5.58}$$

We consider the case where  $|z| < r$ , then

$$\begin{aligned}
2\mu(u_1^c - iu_2^c) = & C_1 \left[ \frac{-1}{2i} \Sigma^* \right] (i\pi z) + C_1 \left( \frac{1}{2i} \right) (-T^*) (-i\pi \bar{z}) + C_2 \left( \frac{1}{4i} \right) T^* (-2i\pi \bar{z}) \\
= & \frac{-(\lambda^* + 3\mu)}{4(\lambda^* + 2\mu)} (\Sigma^*) (z) + \left[ \frac{\lambda^* + 3\mu}{4(\lambda^* + 2\mu)} - \frac{\lambda^* + \mu}{4(\lambda^* + 2\mu)} \right] T^* \bar{z} \\
= & \frac{-(\lambda^* + 3\mu)}{4(\lambda^* + 2\mu)} (\Sigma^*) z + \left[ \frac{\mu}{2(\lambda^* + 2\mu)} \right] T^* \bar{z}.
\end{aligned} \tag{5.59}$$

To find  $\varphi_c$  and  $\psi_c$  we assume they are of the form

$$\varphi_c = A + Cz, \quad \psi_c = B + Dz + Fz^2. \tag{5.60}$$

Since  $2\mu(u_1^c - iu_2^c) = -\bar{z}\varphi_c' + k\overline{\varphi_c} - \psi_c$ , then

$$-\bar{z}\varphi_c' + k\overline{\varphi_c} - \psi_c = \frac{-(\lambda^* + 3\mu)}{4(\lambda^* + 2\mu)} \Sigma^* (z) + \left( \frac{\mu}{2(\lambda^* + 2\mu)} \right) T^* \bar{z}. \tag{5.61}$$

Substituting (5.60) back into (5.61) gives

$$\begin{aligned}
 -C\bar{z} + k(\bar{A} + \bar{C}\bar{z}) - B - D z - F z^2 &= \frac{-(\lambda^* + 3\mu)}{4(\lambda^* + 2\mu)} \Sigma^*(z) + \left[ \frac{\mu}{2(\lambda^* + 2\mu)} \right] T^* \bar{z} \\
 \Rightarrow (k\bar{C} - C)\bar{z} + k\bar{A} - B - D z - F z^2 &= \frac{-(\lambda^* + 3\mu)}{4(\lambda^* + 2\mu)} \Sigma^*(z) + \left[ \frac{\mu}{2(\lambda^* + 2\mu)} \right] T^* \bar{z}
 \end{aligned}$$

Equating the coefficients of  $\bar{z}$  gives

$$\begin{aligned}
 (k\bar{C} - C) &= \left[ \frac{\mu}{2(\lambda^* + 2\mu)} \right] T^* \\
 \text{Re}\{k\bar{C} - C\} &= (k-1)\text{Re}\{C\} = \left[ \frac{\mu T^*}{2(\lambda^* + 2\mu)} \right] \quad (5.62) \\
 \Rightarrow \text{Re}\{C\} &= \frac{\mu T^*}{2(\lambda^* + 2\mu)(k-1)},
 \end{aligned}$$

and

$$\text{Im}\{k\bar{C} - C\} = -(k+1)\text{Im}\{C\} = 0,$$

and hence

$$C = \frac{\mu T^*}{2(\lambda^* + 2\mu)(k-1)}. \quad (5.63)$$

By equating the coefficients of  $z$  we get

$$-D = \frac{-(\lambda^* + 3\mu)}{4(\lambda^* + 2\mu)} \Sigma^* \Rightarrow D = \frac{(\lambda^* + 3\mu)}{4(\lambda^* + 2\mu)} \Sigma^*, \quad (5.64)$$

and

$$K\bar{A} - B = 0 \Rightarrow K\bar{A} = B, F = 0. \quad (5.65)$$

Thus,

$$\begin{aligned} \varphi_c &= A + C z \\ &= A + \frac{\mu T^*}{2(\lambda^* + 2\mu)(k-1)} z, \quad \varphi'_c = \frac{\mu T^*}{2(\lambda^* + 2\mu)(k-1)}, \end{aligned} \quad (5.66)$$

and

$$\begin{aligned} \psi_c &= B + D z + F z^2 \\ &= K\bar{A} + \frac{(\lambda^* + 3\mu)}{4(\lambda^* + 2\mu)} \Sigma^* z. \end{aligned} \quad (5.67)$$

Now we consider the case where  $|z| > r$

$$\begin{aligned}
2\mu(u_1^c - iu_2^c) &= \frac{\lambda^* + 3\mu}{2\pi(\lambda^* + 2\mu)} \left[ \frac{-1}{2i} \Sigma^* \right] \left( \frac{i\pi r^2}{\bar{z}} \right) + \frac{\lambda^* + 3\mu}{2\pi(\lambda^* + 2\mu)} \left( \frac{-1}{2i} T^* \right) \left( \frac{-i\pi r^2}{z} \right) \\
&\quad + \frac{\lambda^* + \mu}{2\pi(\lambda^* + 2\mu)} \left( \frac{1}{4i} T^* \right) \left( \frac{-2i\pi r^2}{z} \right) + \frac{\lambda^* + \mu}{2\pi(\lambda^* + 2\mu)} \frac{(\sigma_{11}^* - \sigma_{22}^* + 2i\sigma_{12}^*)}{4i} \left[ \frac{-2i\pi r^2}{z^2} \left( \bar{z} - \frac{r^2}{z} \right) \right] \\
&= \frac{\lambda^* + 3\mu}{4(\lambda^* + 2\mu)} (-\Sigma^*) \left( \frac{r^2}{\bar{z}} \right) + \left[ \frac{\lambda^* + 3\mu}{4(\lambda^* + 2\mu)} - \frac{\lambda^* + \mu}{4(\lambda^* + 2\mu)} \right] T^* \frac{r^2}{z} - \\
&\quad \frac{\lambda^* + \mu}{(\lambda^* + 2\mu)} \frac{1}{4} (\overline{\Sigma^*}) \left[ \frac{r^2}{z^2} \left( \bar{z} - \frac{r^2}{z} \right) \right] \\
&= \frac{\lambda^* + 3\mu}{4(\lambda^* + 2\mu)} (-\Sigma^*) \frac{r^2}{\bar{z}} + \frac{\mu}{2(\lambda^* + 2\mu)} T^* \frac{r^2}{z} + \frac{-(\lambda^* + \mu)}{4(\lambda^* + 2\mu)} \overline{\Sigma^*} \left[ \frac{r^2}{z^2} \left( \bar{z} - \frac{r^2}{z} \right) \right]. \tag{5.68}
\end{aligned}$$

Since  $2\mu(u_1^c - iu_2^c) = -\bar{z}\varphi_c' + k\overline{\varphi_c} - \psi_c$ , then

$$\begin{aligned}
-\bar{z}\varphi_c' + k\overline{\varphi_c} - \psi_c &= \frac{-(\lambda^* + 3\mu)}{4(\lambda^* + 2\mu)} \Sigma^* \frac{r^2}{\bar{z}} + \frac{\mu}{2(\lambda^* + 2\mu)} T^* \frac{r^2}{z} + \frac{-(\lambda^* + \mu)}{4(\lambda^* + 2\mu)} \overline{\Sigma^*} \left[ \frac{r^2}{z^2} \left( \bar{z} - \frac{r^2}{z} \right) \right] \\
&= \frac{A}{\bar{z}} + \frac{B}{z} + C \frac{\bar{z}}{z^2} + \frac{D}{z^3}, \tag{5.69}
\end{aligned}$$

where

$$\begin{aligned}
A &= \frac{-(\lambda^* + 3\mu)}{4(\lambda^* + 2\mu)} \Sigma^* r^2, B = \frac{\mu}{2(\lambda^* + 2\mu)} T^* r^2, \\
C &= \frac{-(\lambda^* + \mu)}{4(\lambda^* + 2\mu)} \overline{\Sigma^*} r^2 \text{ and } D = \frac{\lambda^* + \mu}{4(\lambda^* + 2\mu)} \overline{\Sigma^*} r^4. \tag{5.70}
\end{aligned}$$

By differentiating equation (5.69) with respect to  $z$ , we obtain

$$\begin{aligned}
 \frac{\partial}{\partial z}(-\bar{z}\phi'_c + k\bar{\phi}_c - \psi_c) &= \frac{\partial}{\partial z}\left(\frac{A}{\bar{z}} + \frac{B}{z} + C\frac{\bar{z}}{z^2} + \frac{D}{z^3}\right) \\
 \Rightarrow -\bar{z}\phi''_c - \psi'_c &= -\frac{B}{z^2} + (-2)C\frac{\bar{z}}{z^3} + (-3)\frac{D}{z^4} \\
 \Rightarrow \phi''_c &= 2C\frac{1}{z^3} \Rightarrow \phi'_c = -\frac{1}{z^2}C \Rightarrow \phi_c = \frac{1}{z}C, -\psi'_c = -\frac{B}{z^2} + (-3)\frac{D}{z^4} \\
 \Rightarrow \psi_c &= -\left(\frac{B}{z} + \frac{D}{z^3}\right).
 \end{aligned} \tag{5.71}$$

By differentiating equation (5.69) with respect to  $\bar{z}$ , we get

$$\begin{aligned}
 \frac{\partial}{\partial \bar{z}}(-\bar{z}\phi'_c + k\bar{\phi}_c - \psi_c) &= \frac{\partial}{\partial \bar{z}}\left(\frac{A}{\bar{z}} + \frac{B}{z} + C\frac{\bar{z}}{z^2} + \frac{D}{z^3}\right) \\
 \Rightarrow -\phi'_c + k\bar{\phi}'_c &= -\frac{A}{(\bar{z})^2} + C\frac{1}{z^2} \Rightarrow k\bar{\phi}'_c = -\frac{A}{(\bar{z})^2} \Rightarrow -k\frac{1}{\bar{z}^2}\bar{C} = -\frac{A}{(\bar{z})^2} \Rightarrow k\bar{C} = A.
 \end{aligned} \tag{5.72}$$

### 5.3 Concluding comments

This chapter has focused on Eshelby's technique for determining the stress, displacement and strain in regions in an infinite elastic body that undergo a change of size or shape. We then translate Eshelby's solution in terms of Muskhelishvili's complex function approach to 2D elasticity, which is more flexible in applications. We will use this approach in Chapter 6 to generalize Eshelby's technique to a matrix with a crack.

## Chapter 6 Eshelby's inclusion with crack

### 6.1 Introduction

In this chapter, we calculate the displacements and stresses for an Eshelby inclusion next to the crack-tip in the 2-dimensional plane stress case. The solution for crack body loaded by point force acting on their face can be calculate by using superposition.

We start with computing the stress field from Eshelby's method without a crack, as shown in Figure 6.1. From this we compute the stress induced along the line where the crack is to be placed.

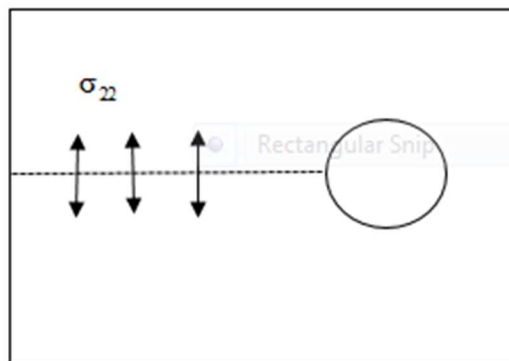


Figure 6.1 Eshelby's inclusion without a crack.

We can then calculate the stresses induced from these forces if as they were acting on the crack flanks, as shown in Figure 6.2.

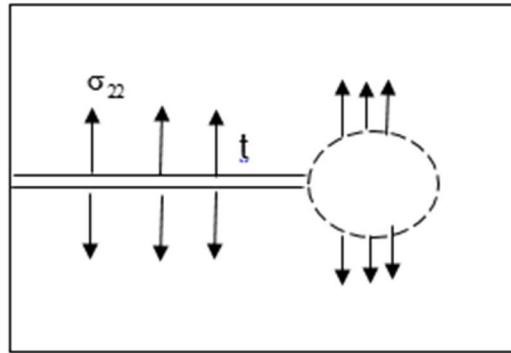


Figure 6.2 the stresses induced from the forces without an Eshelby's inclusion.

Finally, Figure 6.3 shows we can subtract the two solutions to get zero stress on crack flanks, and hence obtain the solution to the equations. The displacement can be calculated from this via the complex potential functions.

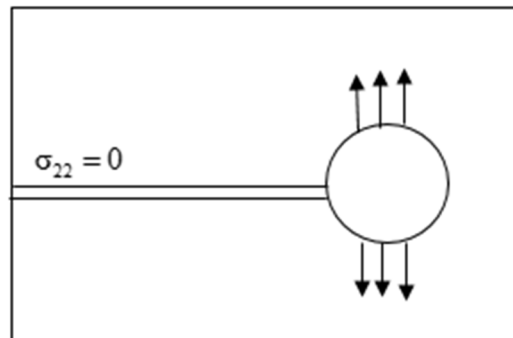


Figure 6.3 the zero stress on crack with Eshelby inclusion

At the end of this chapter, we plot these solutions for some dimensionless units just to validate the model. To illustrate these calculation, we take  $R=2$ ,  $\sigma_{11}^*=-0.3$ ,  $\sigma_{12}^*=0$  and  $\sigma_{22}^*=1$ .



## 6.2 Stresses and displacements on the crack

From the previous chapter, the stresses outside the Eshelby inclusion can be calculated by using the functions:

$$\varphi_c = \frac{A}{z},$$

and

$$\psi_c = -\left[\frac{B}{z} + \frac{C}{z^3}\right],$$

where

$$A = \frac{-(\lambda^* + \mu)}{4(\lambda^* + 2\mu)} (\sigma_{22}^* - \sigma_{11}^* - 2i\sigma_{12}^*) r^2, B = \frac{\mu}{2(\lambda^* + 2\mu)} (\sigma_{22}^* + \sigma_{11}^*) r^2 \text{ and } C = \frac{(\lambda^* + \mu)}{4(\lambda^* + 2\mu)} (\sigma_{22}^* - \sigma_{11}^* - 2i\sigma_{12}^*) r^4.$$

Since

$$\sigma_{11} + \sigma_{22} = 4\text{Re}\{\varphi'\} = 2(\varphi' + \overline{\varphi'}), \quad (6.1)$$

and

$$\sigma_{22} - \sigma_{11} + 2i\sigma_{12} = 2(\bar{z}\varphi'' + \psi'), \quad (6.2)$$

then from (6.1) and (6.2), we get

$$2\sigma_{22} + 2i\sigma_{12} = 2(\varphi' + \bar{\varphi}') + 2(\bar{z}\varphi'' + \psi'), \quad (6.3)$$

This implies that

$$\begin{aligned} \sigma_{22} + i\sigma_{12} &= \varphi' + \bar{\varphi}' + \bar{z}\varphi'' + \psi' \\ &= \frac{-A}{z^2} - \frac{\bar{A}}{\bar{z}^2} + z\left(\frac{2A}{z^3}\right) + \frac{B}{z^2} + 3\frac{C}{z^4}. \end{aligned}$$

Therefore,

$$\begin{aligned} \sigma_{22} &= \text{Re}\left\{\frac{-A}{z^2} - \frac{\bar{A}}{\bar{z}^2} + z\left(\frac{2A}{z^3}\right) + \frac{B}{z^2} + 3\frac{C}{z^4}\right\} \\ \sigma_{11} &= \text{Re}\left\{\frac{-A}{z^2} - \frac{\bar{A}}{\bar{z}^2} - z\left(\frac{2A}{z^3}\right) - \frac{B}{z^2} - 3\frac{C}{z^4}\right\}, \\ \sigma_{12} &= \text{Im}\left\{\frac{-A}{z^2} - \frac{\bar{A}}{\bar{z}^2} + z\left(\frac{2A}{z^3}\right) + \frac{B}{z^2} + 3\frac{C}{z^4}\right\}. \end{aligned} \quad (6.4)$$

Since the inclusion is located in front of a crack tip, centred at  $(R, 0)$  in the Cartesian system with its origin at the crack tip, then the stress everywhere from Eshelby is

$$\sigma_{22} = \text{Re}\left\{\frac{-A}{(z-R)^2} - \frac{\bar{A}}{(\bar{z}-R)^2} + (z-R)\left(\frac{2A}{(z-R)^3}\right) + \frac{B}{(z-R)^2} + \frac{3C}{(z-R)^4}\right\}. \quad (6.5)$$

On the crack, if  $R=r_1+ir_2$  and  $z=a+ib$ , then  $z-R=(a-r_1)+i(b-r_2)=(a-r_1)+i(0)=a-r_1$ . Therefore, the stress on the position where the crack is to be placed from the inclusion is

$$\sigma_{22}=\text{Re}\left\{\frac{-A}{(a-r_1)^2}-\frac{A}{(a-r_1)^2}+\overline{(a-r_1)}\left(\frac{2A}{(a-r_1)^3}\right)+\frac{B}{(a-r_1)^2}+\frac{3C}{(a-r_1)^4}\right\}. \quad (6.6)$$

The displacements outside the inclusion can be calculated from the functions given in Eshelby's method as follows:

$$\begin{aligned} 2\mu(u_1+iu_2) &= -z\bar{\varphi}' + \bar{k}\varphi - \bar{\psi} \\ &= z\left(\frac{\bar{A}}{z^2}\right) + \bar{k}\left(\frac{A}{z}\right) + \frac{\bar{B}}{\bar{z}} + \frac{\bar{C}}{z^3}. \end{aligned} \quad (6.7)$$

Therefore,

$$u_1 = \frac{1}{2\mu} \text{Re}\left\{z\left(\frac{\bar{A}}{z^2}\right) + \bar{k}\frac{A}{z} + \frac{\bar{B}}{\bar{z}} + \frac{\bar{C}}{z^3}\right\}, \quad (6.8)$$

and

$$u_2 = \frac{1}{2\mu} \text{Im}\left\{z\left(\frac{\bar{A}}{z^2}\right) + \bar{k}\frac{A}{z} + \frac{\bar{B}}{\bar{z}} + \frac{\bar{C}}{z^3}\right\}. \quad (6.9)$$

Now, the displacements inside the inclusion can be calculated by using the functions given in Eshelby's method as follows:

$$\varphi = Az \quad \text{and} \quad \psi = Bz,$$

where

$$A = \frac{\mu T^*}{2(\lambda^* + 2\mu)(k-1)}, \quad B = \frac{(\lambda^* + 3\mu)}{4(\lambda^* + 2\mu)} \Sigma^*.$$

The displacement inside the inclusion is therefore given by

$$\begin{aligned} 2\mu(u_1 + iu_2) &= -z\bar{\varphi}' + \bar{k}\varphi - \bar{\psi} \\ &= -z\left(\frac{\mu T^*}{2(\lambda^* + 2\mu)(k-1)}\right) + \bar{k}\left(\frac{\mu T^*}{2(\lambda^* + 2\mu)(k-1)}\right)z - \frac{(\lambda^* + 3\mu)}{4(\lambda^* + 2\mu)} \Sigma^* \bar{z}. \end{aligned}$$

This implies that

$$u_1 + iu_2 = \frac{-T^* z}{4(\lambda^* + 2\mu)(k-1)} + \frac{\bar{k} T^*}{4(\lambda^* + 2\mu)(k-1)} z - \frac{(\lambda^* + 3\mu)}{8(\lambda^* + 2\mu)} \Sigma^* \bar{z}, \quad (6.10)$$

and hence

$$u_1 = \text{Re}\{A_1 z + B_1 \bar{k} z - C_1 \bar{z}\}, \quad (6.11)$$

and

$$u_2 = \text{Im}\{A_1 z + B_1 \bar{k} z - C_1 \bar{z}\}, \quad (6.12)$$

where  $A_1 = \frac{-(\sigma_{11}^* + \sigma_{22}^*)}{4(\lambda^* + 2\mu)(k-1)}$ ,  $B_1 = \frac{(\sigma_{11}^* + \sigma_{22}^*)}{4(\lambda^* + 2\mu)(k-1)}$  and  $C_1 = \frac{(\lambda^* + 3\mu)}{8(\lambda^* + 2\mu)\mu} (\sigma_{22}^* - \sigma_{11}^* - 2i\sigma_{12}^*)$ .

### 6.3 The stresses and the displacements generated by a Westergaard function, $Z$ , for mode I problems

We define a Westergaard function,  $Z_I$ , for Mode I problems by

$$Z_I(t, z) = \frac{P}{\pi(z-t)} \sqrt{\frac{t}{z}}. \quad (6.13)$$

By integrating Westergaard's function,  $Z_I$ , using thus expression for  $\sigma_{22}$  above as the force on the crack, one can get the stress  $\sigma_{22}$  everywhere corresponding to  $\sigma_{22}$  along the crack. Thus, we take

$$Z = \int_{-\infty}^0 \frac{\sigma_{22}}{\pi(z-t)} \sqrt{\frac{t}{z}} dt, \quad t < 0. \quad (6.14)$$

Since, from (6.5),  $\sigma_{22}(t) = \frac{B}{(t-r_1)^2} + \frac{3C}{(t-r_1)^4}$ , then

$$\sigma_{22}(t_1) = \frac{B}{(-t_1-r_1)^2} + \frac{3C}{(-t_1-r_1)^4} = \frac{B}{(t_1+r_1)^2} + \frac{3C}{(t_1+r_1)^4}, \quad (6.15)$$

and

$$Z = \int_0^{\infty} \left[ \frac{B}{(t_1+r_1)^2} + \frac{3C}{(t_1+r_1)^4} \right] \frac{1}{\pi(z+t_1)} \sqrt{\frac{t_1}{z}} dt_1. \quad (6.16)$$

From (6.16) we start to find the integral

$$\int_0^{\infty} \frac{B}{(t_1+r_1)^2} \frac{1}{\pi(z+t_1)} \sqrt{\frac{t_1}{z}} dt_1 = \frac{B}{\pi\sqrt{z}} \int_0^{\infty} \frac{\sqrt{t_1}}{(z+t_1)(t_1+r_1)^2} dt_1. \quad (6.17)$$

Now,

$$\int_0^{\infty} \frac{\sqrt{t_1}}{(z+t_1)(t_1+r_1)^2} dt_1.$$

Using cantor integration by taking the cantor:

$$\gamma = [e, R] - C_e + [R, e] + C_R.$$

Therefore,

$$\begin{aligned} \oint_{\gamma} f(t_1) dt_1 &= 2i\pi \operatorname{Res}_{t_1=-z, -r_1} f(t_1) \\ &= 2i\pi \left\{ \lim_{t_1 \rightarrow -z} \frac{\sqrt{t_1} (t_1+z)}{(t_1+r_1)^2 (t_1+z)} \right\} + \lim_{t_1 \rightarrow -r_1} \frac{d}{dt_1} \left( \frac{\sqrt{t_1} (t_1+r_1)^2}{(t_1+r_1)^2 (t_1+z)} \right) \} \\ &= 2i\pi \left\{ \frac{\sqrt{-z}}{(-z+r_1)^2} + \lim_{t_1 \rightarrow -r_1} \frac{d}{dt_1} \left( \frac{\sqrt{t_1}}{(z+t_1)} \right) \right\} \\ &= 2i\pi \left\{ \frac{i\sqrt{z}}{(r_1-z)^2} + \frac{(z+r_1)}{2(z-r_1)^2 \sqrt{-r_1}} \right\} \\ &= 2i\pi \left\{ \frac{i\sqrt{z}}{(r_1-z)^2} + \frac{(z+r_1)}{2i(r_1-z)^2 \sqrt{r_1}} \right\} = \frac{-2\pi\sqrt{z}}{(r_1-z)^2} + \frac{\pi(z+r_1)}{(r_1-z)^2 \sqrt{r_1}}, \end{aligned}$$

then

$$\begin{aligned} \frac{B}{\pi\sqrt{z}} \int_0^\infty \frac{\sqrt{t_1}}{(z+t_1)(t_1+r_1)^2} dt_1 &= \frac{B}{\pi\sqrt{z}} \left[ \frac{-2\pi\sqrt{z}}{(r_1-z)^2} + \frac{\pi(z+r_1)}{(r_1-z)^2\sqrt{r_1}} \right] \\ &= B \left[ \frac{-2}{(r_1-z)^2} + \frac{(z+r_1)}{\sqrt{z}\sqrt{r_1}(r_1-z)^2} \right]. \end{aligned} \quad (6.18)$$

Let  $z=-u$ . Then

$$\begin{aligned} \frac{B}{\pi\sqrt{z}} \int_0^\infty \frac{\sqrt{t_1}}{(z+t_1)(t_1+r_1)^2} dt_1 &= B \left[ \frac{-1}{(u+r_1)^2} + \frac{(-u+r_1)}{2\sqrt{-u}\sqrt{r_1}(u+r_1)^2} \right] \\ &= B \left[ \frac{-1}{(u+r_1)^2} - i \frac{(-u+r_1)}{2\sqrt{u}\sqrt{r_1}(u+r_1)^2} \right]. \end{aligned} \quad (6.19)$$

Since  $t_1=u$ , then  $t=-u$  and we can write equation (6.19) as follows

$$\begin{aligned} \frac{B}{\pi\sqrt{z}} \int_0^\infty \frac{\sqrt{t_1}}{(z+t_1)(t_1+r_1)^2} dt_1 &= B \left[ \frac{-1}{(u+r_1)^2} + \frac{(-u+r_1)}{2\sqrt{-u}\sqrt{r_1}(u+r_1)^2} \right] \\ &= B \left[ \frac{-1}{(u+r_1)^2} - i \frac{(-u+r_1)}{2\sqrt{u}\sqrt{r_1}(u+r_1)^2} \right] \\ &= B \left[ \frac{-1}{(r_1-t)^2} - i \frac{(t+r_1)}{2\sqrt{-t}\sqrt{r_1}(-t+r_1)^2} \right]. \end{aligned} \quad (6.20)$$

Therefore,

$$\operatorname{Re} \left\{ \frac{1}{(r_1-t)^2} + i \frac{(t+r_1)\sqrt{u}}{2u(t-r_1)^2\sqrt{r_1}} \right\} = \frac{1}{(-u-r_1)^2} = \frac{1}{(t-r_1)^2}, \quad t=-u. \quad (6.21)$$

Using a similar method to the previous integration, we can find the integral

$$\int_{-\infty}^0 \frac{3C\sqrt{-t}}{\pi\sqrt{z}(z-t)(t-r_1)^4} \sqrt{z} dt. \quad (6.22)$$

That is, we calculate

$$\begin{aligned} \oint_{\gamma} f(t) dt &= 2i\pi \operatorname{Res}_{t=r_1} f(t) \\ &= 2i\pi \left\{ \lim_{t \rightarrow z} \frac{\sqrt{-t} (t-z)}{\pi(t-r_1)^4 (z-t)\sqrt{z}} \right\} + \lim_{t \rightarrow r_1} \frac{1}{3!} \frac{d^3}{dt^3} \left( \frac{\sqrt{-t} (t-r_1)^4}{\pi(t-r_1)^4 (z-t)\sqrt{z}} \right) \\ &= 2i\pi \left\{ \left[ \frac{-i\sqrt{z}}{\pi(z-r_1)^4 \sqrt{z}} \right] + \lim_{t \rightarrow r_1} \frac{1}{3!} \frac{d^3}{dt^3} \left( \frac{i\sqrt{t}}{\pi(z-t)\sqrt{z}} \right) \right\} \\ &= \left[ \frac{2}{(z-r_1)^4} \right] + \frac{2i}{6} \left( \frac{-i}{\sqrt{z}} \right) \frac{(-3)(5r_1^3 + 15r_1^2 z - 5r_1 z^2 + z^3)}{8r_1^2 (r_1 - z)^4} \\ &= \frac{2}{(z-r_1)^4} - \frac{(5r_1^3 + 15r_1^2 z - 5r_1 z^2 + z^3)}{8\sqrt{z} r_1^2 (r_1 - z)^4}. \end{aligned} \quad (6.23)$$

Therefore,

$$\int_{-\infty}^0 \frac{3C\sqrt{-t}}{\pi\sqrt{z}(z-t)(t-r_1)^4} \sqrt{z} dt = 3C \left[ \frac{2}{(z-r_1)^4} - \frac{(5r_1^3 + 15r_1^2 z - 5r_1 z^2 + z^3)}{8\sqrt{z} r_1^2 (r_1 - z)^4} \right]. \quad (6.24)$$

Let  $z = -u$ . Then



$$\begin{aligned}
\oint_{\gamma} f(t) dt &= \frac{2}{(-u-r_1)^4} - \frac{(5r_1^3 + 15r_1^2(-u) - 5r_1(-u)^2 + (-u)^3)}{8r_1^{\frac{5}{2}}(r_1+u)^4 \sqrt{-u}} \\
&= \frac{2}{(-u-r_1)^4} + i \frac{(5r_1^3 - 15r_1^2u - 5r_1u^2 - u^3)}{8r_1^{\frac{5}{2}}(r_1+u)^4 \sqrt{u}}.
\end{aligned} \tag{6.25}$$

Therefore,

$$\operatorname{Re} \left\{ \frac{1}{(-u-r_1)^4} + i \frac{(5r_1^3 - 15r_1^2u - 5r_1u^2 - u^3)}{16r_1^{\frac{5}{2}}(r_1+u)^4 \sqrt{u}} \right\} = \frac{1}{(-u-r_1)^4} = \frac{1}{(t-r_1)^4}, t=-u. \tag{6.26}$$

From (6.16), (6.18) and (6.24), we get

$$Z = B \left[ \frac{-2}{(r_1-z)^2} + \frac{(z+r_1)}{\sqrt{z}\sqrt{r_1}(r_1-z)^2} \right] + 3C \left[ \frac{2}{(z-r_1)^4} - \frac{(5r_1^3 + 15r_1^2z - 5r_1z^2 + z^3)}{8\sqrt{z}r_1^{\frac{5}{2}}(r_1-z)^4} \right]. \tag{6.27}$$

Recall that  $\psi' = -z\phi''$ , hence  $\psi = \int -z\phi'' dz = -z\phi' + \phi$ . We now define  $M, N$ , and  $L$  by

$$M = \phi' = \frac{1}{2} Z, \tag{6.28}$$

$$N = \phi'' = \frac{1}{2} Z', \tag{6.29}$$

and

$$L = \phi = \int \phi' dz. \tag{6.30}$$

From (6.3), the stress generated by a Westergaard function,  $Z$ , for mode I problems is

$$\begin{aligned}\sigma_{22} &= \text{Re}\{\phi' + \bar{\phi}' + z\phi'' + \bar{z}\bar{\phi}'' + \psi'\} \\ &= \text{Re}\{M + \bar{M} + zN + \bar{z}\bar{N}\}.\end{aligned}\quad (6.31)$$

From (6.7), the displacement generated by a Westergaard function,  $Z$ , for mode I problems is

$$\begin{aligned}2\mu(u_1 + iu_2) &= -z\bar{\phi}' + \bar{z}\phi' - \bar{\psi} \\ &= -z\bar{M} + \bar{z}L + \bar{z}\bar{M} - \bar{L}.\end{aligned}\quad (6.32)$$

Therefore,

$$u_1 = \frac{1}{2\mu} \text{Re}\{-z\bar{M} + \bar{z}L + \bar{z}\bar{M} - \bar{L}\}, \quad (6.33)$$

and

$$u_2 = \frac{1}{2\mu} \text{Im}\{-z\bar{M} + \bar{z}L + \bar{z}\bar{M} - \bar{L}\}. \quad (6.34)$$

We now plot the displacements for the case where  $R=0.5\text{mm}$ ,  $\sigma_{11}^*=-30\text{ MPa}$ ,  $\sigma_{12}^*=0$  and  $\sigma_{22}^*=100\text{MPa}$ .

The values above were chosen to be of typical magnitudes, but the exact values required to accurately match the model against experimental data will require further investigation. We comment on this in Section 7.

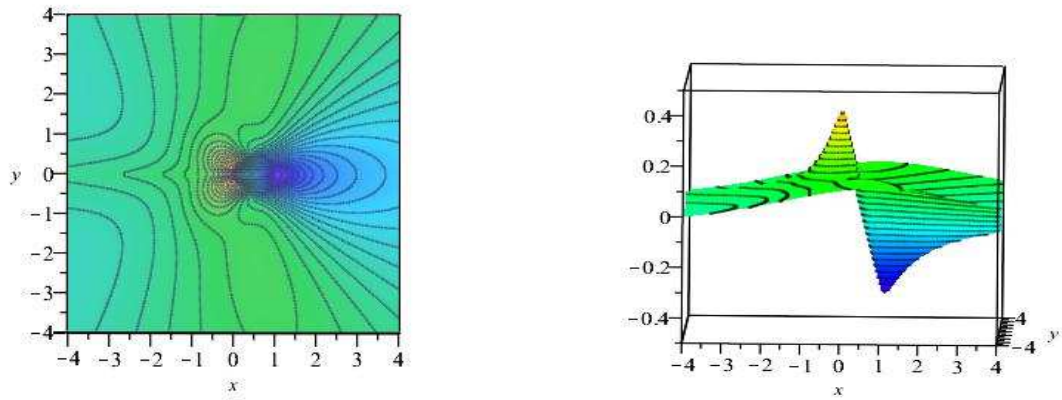


Figure 6.4 The x-displacement of subtract Eshelby inclusion from displacements generated by a Westergaard function,  $Z$ , with Lamé constants for aluminum ( $\lambda^*=53.5$  GPa,  $\mu=26.6$  GPa) and  $R=0.5$ mm,  $\sigma_{11}^*=-30$  MPa,  $\sigma_{12}^*=0$  and  $\sigma_{22}^*=100$ MPa.

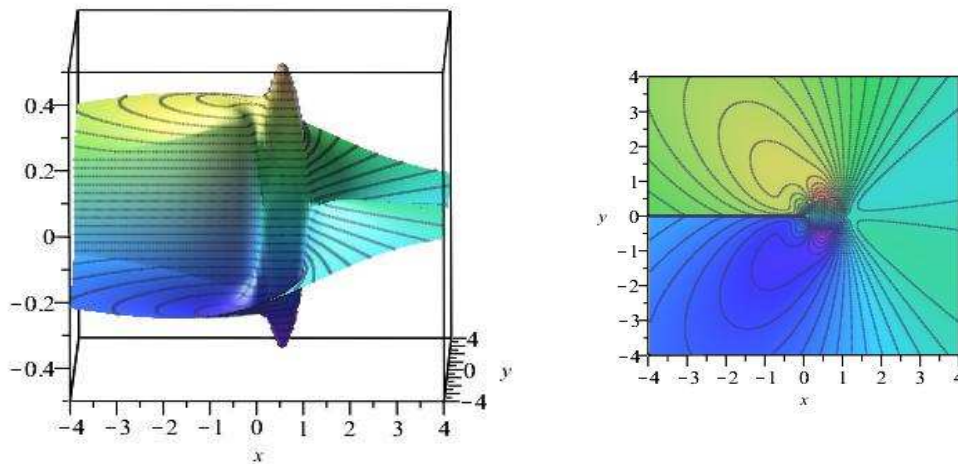


Figure 6.5 The y-displacement of subtract Eshelby inclusion from displacements generated by a Westergaard function,  $Z$ , with Lamé constants for aluminium ( $\lambda^*=53.5$  GPa,  $\mu=26.6$  GPa) and  $R=0.5$ mm,  $\sigma_{11}^*=-30$  MPa,  $\sigma_{12}^*=0$  and  $\sigma_{22}^*=100$ MPa.

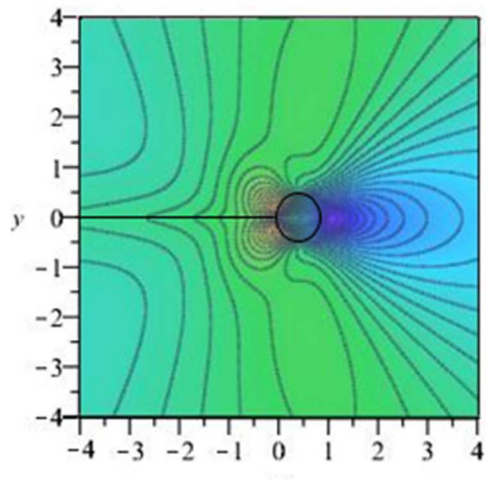


Figure 6.6 As in Figure 6.4, but with inclusion and crack highlighted

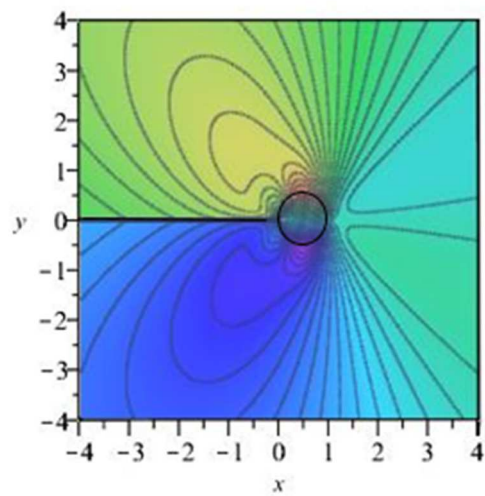


Figure 6.7 As in Figure 6.5, but with inclusion and crack highlighted

## 6.4 Concluding comments

We computed the stress induced along the line where the crack is to be placed by using the stress field from Eshelby's method without a crack. And then calculated the stresses induced from the forces (stress) as if they were acting on the crack flank. After that, we got the solution with zero stress on crack flank by subtracting the two solutions. This work includes illustrations of these calculation using the software Maple.

## Chapter 7 Summation of concluding comments

### 7.1 Conclusions of thesis

In this thesis, we calculated the stresses and displacement for an Eshelby inclusion where the matrix includes a crack. To do this we transformed Eshelby's solution in terms of Muskhelishvili's complex potential functions for 2D elasticity. We also have developed tools to tackle more general problems for future investigation.

We have concentrated only on one particular case where the inclusion is a disc in front of the crack just touching the crack tip, depend on four parameters ( $R, \sigma_{11}^*, \sigma_{22}^*, \sigma_{12}^*$ ). Here we want to discuss the limitations and validation of this model and make suggestions for future work.

### 7.2 Limitations

We have only considered a circle inclusion. It is known that plastic region ahead of crack tip has a non-circle shape, but the approximation of a circle is probably accurate enough for simplified model such as we consider here an initial approximate.

We used uniform  $\sigma_{ij}^*$ , but non uniform  $\sigma_{ij}^*$  would be more realistic.

In this model, we have not include crack flanks, but there are research works such as [42] would suggest that closure/ shield effect is more clearly related to the behaviour at crack tip rather than flanks.

### **7.3 Validation**

Visually, this model looks good for typical parameter. No external forces is applied in the model and if we added an external forces would dominate terms from model. New experiment for uncracked specimen being planned to investigate this in more details.

### **7.4 Suggestions for future work**

According to the work done in this thesis, there are still some issues that could be addressed in future work such as the following:

1. The inclusion near the crack-tip in the 2-dimensional plane stress may have various shapes and non-linear eigenstrains. The tools for these have been developed in this thesis.
2. This work should be validated by experimental work to quantify the effectiveness of the model in relation to different specimen geometries and fatigue situations.
3. Once validated, the model could be used to estimate residual forces and plastic zone size due to the crack and hence provide an alternative set of parameters to describe fatigue crack closure.
4. Comparison of the model of the plastic region of a crack developed here with the CJP approach. It would be interesting to see what could be understood from the parameters of each model and how they could be related.

5. Extend the work on Westergaard's function in chapter 4, to define the relationship between forces on the crack flank with the full field displacement data. Such a relationship could then be inverted so that displacement data could be used to predict the equivalent forces which placed along the crack flanks of a perfect crack would best generate the full field data.
6. We have plotted the solution of this model for mode I, but could be developed for mixed loading conditions by assuming  $\sigma_{12}^* \neq 0$ .



## References

- [1] Suresh, S., 1998. *Fatigue of materials*. Cambridge university press.
- [2] Lu, Y., 2011. *Full-field modelling of crack tip shielding phenomena* (Doctoral dissertation, University of Plymouth).
- [3] Eshelby, J.D., 1957, August. The determination of the elastic field of an ellipsoidal inclusion, and related problems. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences* (Vol. 241, No. 1226, pp. 376-396). The Royal Society.
- [4] Weinberger, C., Cai, W. and Barnett, D., 2005. Lecture Notes-Elasticity of Microscopic Structures. *ME340-Stanford University*.
- [5] Irwin, G.R. and Fracture, S.F., 1958. Encyclopedia of Physics. *Springer Verlag, Berlin*, pp.551-590.
- [6] Hou, C.Y., 1994. *A crack-closure model for the fatigue behavior of notched components and welded joints* (Doctoral dissertation, University of Illinois at Urbana-Champaign).
- [7] Elber, W. (1970) Fatigue crack closure under cyclic tension. *Engineering Fracture Mechanics*, 2(1), pp.37-45.
- [8] Anderson, T.L. (2005). Fracture Mechanics: Fundamentals and Applications, *CRC Press LLC*, USA.
- [9] Pippan, R. and Hohenwarter, A., 2017. Fatigue crack closure: a review of the physical phenomena. *Fatigue & fracture of engineering materials & structures*, 40(4), pp.471-495.

- [10] Renshaw, J., Holland, S.D. and Thompson, R.B., 2008. Measurement of crack opening stresses and crack closure stress profiles from heat generation in vibrating cracks. *Applied Physics Letters*, 93(8), p.081914.
- [11] Janssen, M., Zuidema, J. and Wanhill, R.J.H., 2006. Fracture mechanics VSSD.
- [12] James, M.N., Christopher, C.J., Lu, Y. and Patterson, E.A., 2013. Local crack plasticity and its influences on the global elastic stress field. *International Journal of Fatigue*, 46, pp.4-15.
- [13] Muskhelishvili, N.I., 1977. *Some basic problems of the mathematical theory of elasticity*. Noordhoff International Publishing, Groningen.
- [14] Christopher, C.J., James, M.N., Patterson, E.A. and Tee, K.F. (2007). Towards a new model of crack tip stress fields. *Int. J. Fract.*, 148, 361-371.
- [15] Nowell, D., 2007. Techniques for experimental measurement of fatigue crack closure. In *Applied Mechanics and Materials* (Vol. 7, pp. 3-9). Trans Tech Publications.
- [16] Elber, W., 1971. The significance of fatigue crack closure. In *Damage tolerance in aircraft structures*. ASTM International.
- [17] Riddell, P.J., 1995. Electrical resistance variations across a growing fatigue crack as a function of the load cycle. *Fatigue & Fracture of Engineering Materials & Structures*, 18(1), pp.157-166.
- [18] Walker, N. and Beevers, C.J., 1979. A fatigue crack closure mechanism in titanium. *Fatigue & Fracture of Engineering Materials & Structures*, 1(1), pp.135-148.
- [19] Stewart, A.T., 1980. The influence of environment and stress ratio on fatigue crack growth at near threshold stress intensities in low-alloy steels. *Engineering Fracture Mechanics*, 13(3), pp.463-478.

- [20] Ritchie, 7.R., Suresh, S. and Moss, C.M., 1980. Near-threshold fatigue crack growth in 2 1/4 Cr-1Mo pressure vessel steel in air and hydrogen. *Journal of Engineering Materials and Technology*, 102(3), pp.293-299.
- [21] Ohta, A., Kosuge, M. and Sasaki, E., 1978. Fatigue crack closure over the range of stress ratios from– 1 to 0.8 down to stress intensity threshold level in HT80 steel and SUS304 stainless steel. *International Journal of Fracture*, 14(3), pp.251-264.
- [22] Zhu, M.L., Xuan, F.Z. and Tu, S.T., 2015. Effect of load ratio on fatigue crack growth in the near-threshold regime: A literature review, and a combined crack closure and driving force approach. *Engineering Fracture Mechanics*, 141, pp.57-77.
- [23] Suresh, S., Parks, D.M. and Ritchie, R.O., 1981. Fatigue thresholds. In *Proceedings of 1st International Conference, Stockholm, J. Backbond, A. Blom, and C J. Beevers, eds., EMAS Pubi. Ltd., Warley, UK* (pp. 391-408).
- [24] Budiansky, B. and Hutchinson, J.V., 1977. Analysis of Closure in Fatigue.
- [25] Dugdale, D.S., 1960. Yielding of steel sheets containing slits. *Journal of the Mechanics and Physics of Solids*, 8(2), pp.100-108.
- [26] Mirzaei, M. and Provan, J.W., 1992. A new method for the analysis and assessment of fatigue crack closure. I: modeling. *Theoretical and applied fracture mechanics*, 18(1), pp.47-58.
- [27] Mirzaei, M. and Provan, J.W., 1992. A new method for the analysis and assessment of fatigue crack closure. I: modelling. *Theoretical and applied fracture mechanics*, 18(1), pp.59-63.
- [28] Newman, J.C., 1976. A finite-element analysis of fatigue crack closure. In *Mechanics of crack growth*. ASTM International.
- [29] Fleck, N.A. and Newman, J.C., 1988. Analysis of crack closure under plane strain conditions. In *Mechanics of fatigue crack closure*. ASTM International.

- [30] Sun, W., 1992. *Finite element simulations of fatigue crack growth and closure* (Doctoral dissertation, University of Illinois at Urbana-Champaign).
- [31] Newman, J.C., 1981. A crack-closure model for predicting fatigue crack growth under aircraft spectrum loading. In *Methods and models for predicting fatigue crack growth under random loading*. ASTM International.
- [32] Newman Jr, J.C., 1983. *A nonlinear fracture mechanics approach to the growth of small cracks*. NATIONAL AERONAUTICS AND SPACE ADMINISTRATION HAMPTON VA LANGLEY RESEARCH CENTER.
- [33] Nowell, D. and De Matos, P.F.P., 2010. Application of digital image correlation to the investigation of crack closure following overloads. *Procedia Engineering*, 2(1), pp.1035-1043.
- [34] Wei, L.W. and James, M.N., 2000. A study of fatigue crack closure in polycarbonate CT specimens. *Engineering Fracture Mechanics*, 66(3), pp.223-242.
- [35] Pacey, M.N., James, M.N. and Patterson, E.A., 2005. A new photoelastic model for studying fatigue crack closure. *Experimental Mechanics*, 45(1), pp.42-52.
- [36] James, M.N., 1996. Some unresolved issues with fatigue crack closure-measurement, mechanism and interpretation problems. In *ICF 9-Sydney, Australia-1997*.
- [37] Irwin GR (1960) Plastic zone near a crack and fracture toughness. *Proc Seventh Sagamore Ordnance Mater Res Conf IV*: 63-78.
- [38] Zhu, P., Yang, L., Li, Z. and Sun, J., 2010. The shielding effects of the crack-tip plastic zone. *International journal of fracture*, 161(2), pp.131-139.
- [39] Berto, F. and Lazzarin, P., 2010. On higher order terms in the crack tip stress field. *International Journal of Fracture*, 161(2), pp.221-226.

- [40] Vasco-Olmo, J.M., James, M.N., Christopher, C.J., Patterson, E.A. and Díaz, F.A., 2016. Assessment of crack tip plastic zone size and shape and its influence on crack tip shielding. *Fatigue & Fracture of Engineering Materials & Structures*, 39(8), pp.969-981.
- [41] Christopher, C.J., Laboviciute, G., James, M.N. and Patterson, E.A., 2013. Extension of the CJP model to mixed mode I and mode II. *Frattura ed Integrità Strutturale*, 7(25), pp.161-166.
- [42] Christopher, C.J., James, M.N., Patterson, E.A. and Tee, K.F., 2008. A quantitative evaluation of fatigue crack shielding forces using photoelasticity. *Engineering Fracture Mechanics*, 75(14), pp.4190-4199.
- [43] Mura, T., 1987. General theory of eigenstrains. In *Micromechanics of defects in solids* (pp. 1-73). Springer, Dordrecht.
- [44] Rudnicki, J.W., 2011. Eshelby's technique for analyzing inhomogeneities in geomechanics. In *Mechanics of crustal rocks* (pp. 43-72). Springer Vienna.
- [45] Taya, M. and Chou, T.W., 1981. On two kinds of ellipsoidal inhomogeneities in an infinite elastic body: an application to a hybrid composite. *International Journal of Solids and Structures*, 17(6), pp.553-563.
- [46] Withers DJ, Stobbs WM, Pederson OB (1989) The application of the Eshelby method of internal stress determination to short fibre metal matrix composites. *Acta Metall* 37:3061-3084
- [47] Johnson, W.C., Earmme, Y.Y. and Lee, J.K., 1980. Approximation of the Strain Field Associated With an Inhomogeneous Precipitate—Part 2: The Cuboidal Inhomogeneity. *Journal of Applied Mechanics*, 47(4), pp.781-788.
- [48] Slaughter, W.S., 2001. The Linearized Theory of Elasticity, Section 9.4. *Birkhausen, Boston*.

- [49] Li, Z. and Chen, Q., 2002. Crack-inclusion interaction for mode I crack analyzed by Eshelby equivalent inclusion method. *International Journal of Fracture*, 118(1), pp.29-40.
- [50] Zou, W.N., He, Q.C. and Zheng, Q.S., 2012. Inclusions in a finite elastic body. *International Journal of Solids and Structures*, 49(13), pp.1627-1636.
- [51] Duan, J. and Li, Z., 2002. Tensile stresses in an inclusion ahead of the crack tip. *International journal of fracture*, 115(4), pp.75-80.
- [52] Schijve, J., 1988. Fatigue crack closure: observations and technical significance. In *Mechanics of fatigue crack closure*. ASTM International.
- [53] Jiang, S., Zhang, W., Li, X. and Sun, F., 2014. An analytical model for fatigue crack propagation prediction with overload effect. *Mathematical Problems in Engineering*, 2014.
- [54] Sun, C. T. and Jin, Z. H., 2012. Fracture Mechanics. Academic press.
- [55] Chan, K.S., 2013. Growth Characteristics of Large Fatigue Cracks. *Encyclopedia of Tribology*, pp.1580-1592.
- [56] Challis, L. and Sheard, F., 2003. The green of green functions. *Physics Today*, 56(12), pp.41-46.
- [57] Eshelby, J.D., 1961. Progress in solid mechanics: Vol. 1. Edited by IN Sneddon and R. Hill, North-Holland Publishing Company, Amsterdam, 1960, 448 pp., 100s.
- [58] Patricio, M. and Mattheij, R., 2007. Crack propagation analysis. *CASA report*, pp.07-03.
- [59] Zehnder, A.T., 2007. Lecture notes on fracture mechanics. Available for public use at Cornell University website [http://ecommons.library.cornell.edu/bitstream/1813/3075/6/fracture\\_notes\\_2008.pdf](http://ecommons.library.cornell.edu/bitstream/1813/3075/6/fracture_notes_2008.pdf).

- [60] Ogura, K. and Ohji, K. (1977) FEM analysis of crack closure and delay effect in fatigue crack growth under variable amplitude loading. *Engng Fract. Mech.* **9**, 471-480.
- [61] Marci, G. and Packman, P.F., 1980. The effects of the plastic wake zone on the conditions for fatigue crack propagation. *International Journal of Fracture*, **16**(2), pp.133-153.
- [62] Kuykendall, W.P., Cash, W.D., Barnett, D.M. and Cai, W., 2012. On the existence of Eshelby's equivalent ellipsoidal inclusion solution. *Mathematics and Mechanics of Solids*, **17**(8), pp.840-847.
- [63] Correia, J.A., De Jesus, A.M., Moreira, P.M. and Tavares, P.J., 2016. Crack closure effects on fatigue crack propagation rates: application of a proposed theoretical model. *Advances in Materials Science and Engineering*, 2016.
- [64] Hou, C.Y. and Charng, J.J., 1996. Estimation of plasticity-induced crack closure in a pre-existing plastic zone. *International journal of fatigue*, **18**(7), pp.463-474.
- [65] Vasudeven, A.K., Sadananda, K. and Louat, N., 1994. A review of crack closure, fatigue crack threshold and related phenomena. *Materials Science and Engineering: A*, **188**(1-2), pp.1-22.
- [66] James, M.N. and Smith, G.C., 1983. Crack closure and surface microcrack thresholds—some experimental observations. *International Journal of fatigue*, **5**(2), pp.75-78.
- [67] Bouledroua, O., Meliani, M.H. and Pluvinage, G., 2016. A review of T-stress calculation methods in fracture mechanics computation. *Nature & Technology*, (15), p.20.